UNIVERSIDADE FEDERAL RURAL DE PERNAMBUCO
PRÓ-REITORIA DE PESQUISA E PÓS-GRADUAÇÃO
PROGRAMA DE PÓS-GRADUAÇÃO EM BIOMETRIA E ESTATÍSTICA APLICADA

Daniel Leonardo Ramírez Orozco

Some Results on Stochastic Comparisons by Majorization Theory and Goodness-of-Fit Measures Based on The Mellin Transform

Recife-PE
04 de Novembro de 2022

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Concentration Field: Biometry and Applied Statistics

Advisor: Prof. Dr. Frank Sinatra Gomes da Silva

Co-Advisor: Prof. Dr. Josimar Mendes de Vasconcelos

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Contents

Nomenclature

- EE: Exponential Exponentiated.
- EG: Exponentiated Generalized.
- EGF: Exponentiated Generalized Fréchet.
- EGN: Exponentiated Generalized Normal.
- EGBS: Exponentiated Generalized Birnbaum-Saunders.
- EGGu: Exponentiated Generalized Gumbel.
- EGNH: Exponentiated Generalized Nadarajah-Haghighi.
- EGKw: Exponentiated Generalized Kumaraswamy.
- $X_i \stackrel{st}{=} X$: X_i is equal stochastically to X.
- $\boldsymbol{\alpha} \stackrel{m}{\succeq} \boldsymbol{\beta}$: $\boldsymbol{\alpha}$ is said to majorize $\boldsymbol{\beta}$.
- $\boldsymbol{\alpha} \stackrel{w}{\succeq} \boldsymbol{\beta}$: $\boldsymbol{\alpha}$ is said to weakly supermajorize $\boldsymbol{\beta}$.
- $\boldsymbol{\alpha} \succeq_w \boldsymbol{\beta}$: $\boldsymbol{\alpha}$ is said to weakly submajorize $\boldsymbol{\beta}$.
- $X \leq_{lr} Y$: likelihood ratio order.
- $X \leq_{hr} Y$: hazard rate order.
- $X \leq_{rhr} Y$: reversed hazard rate order.
- $\stackrel{sign}{=}$: equals sign.
- TIW: Transmuted Inverse Weibull.
- CDF: Cumulative distribution function.
- PDF: Probability density function.
- GoF: Goodness-of-Fit.
- ML: Maximum Likelihood.
- LC: Log-Cumulants.
- LM: Log-Moments.
- MT: Mellin transform.
- SD: Standard Deviation.
- CV: Coefficient of Variation.

Abstract

In this thesis, we provide results in two fields. In the first one, the main idea is to make stochastic comparisons by using the majorization theory. We investigate conditions on the parameters of the Exponentiated Generalized class with equal and different baselines. In the second one, to describe the fits of models to the data by using Goodness-of-Fit measures, we use the Transmuted Inverse Weibull distribution and derive the Mellin Transform from that model. We estimate the parameters by Moments, Maximum Likelihood function, and Log-Cumulants methods. An interesting analysis is carried out with Hotelling's T^2 Statistic. In this step, we construct Log-Cumulants diagrams and furnish ellipses of confidence for Log-Cumulants. The applications in this part were based on lifetime datasets such as mechanical components' survival time, electrical insulator films' failure times, and censored data from times of bladder cancer patients. We present some simulation results to verify the performance of the Moments, Maximum Likelihood function, and Log-Cumulants methods. Kolmogorov-Smirnov and Cramér-von Mises as Goodness-of-Fit criterias are used. Monte Carlo methods such as bootstrap and Jackknife are used to estimate population characteristics. The Mellin-based Goodness-of-Fit test's performance is compared with Anderson-Darling and Kolmogorov-Smirnov tests.

Keywords: Goodness-of-Fit, Hotelling's T^2 Statistic, Log-Cumulants, Majorization, Order statistics, Parameters Estimation.

Resumen

En esta tesis, se aportan resultados en dos campos. En el primero, la idea principal es realizar comparaciones estocásticas utilizando la teoría de la mayorización. Se investigan condiciones sobre los parámetros de la clase exponencial generalizada con distribuciones-base iguales y diferentes. En el segundo campo, se describen los ajustes de los modelos a los datos utilizando medidas de bondad-de-ajuste, se utiliza la distribución Weibull Inversa Transmutada y se deriva la transformada de Mellin de ese modelo. Se estiman los parámetros mediante los métodos de Momentos, Función de Máxima Verosimilitud y Log-Cumulantes. Un análisis interesante se lleva a cabo con el estadístico T^2 de Hotelling. En lo que sigue, se construyen diagramas de Log-Cumulantes y proporcionamos elipses de confianza para Log-Cumulantes. Las aplicaciones de esta parte se basan en conjuntos de datos de vida útil, como el tiempo de supervivencia de los componentes mecánicos, los tiempos de falla de las películas aislantes eléctricas y los datos censurados de los tiempos de los pacientes con cáncer de vejiga. Presentamos algunos resultados de simulación para verificar el rendimiento de los métodos de Momentos, Función de Máxima Verosimilitud y Log-Cumulantes. Se utilizan los criterios de bondad de ajuste de Kolmogorov-Smirnov y Cramér-von Mises. Para estimar las características de la población se utilizan métodos de Monte Carlo como bootstrap y Jackknife. El rendimiento de la prueba de bondad de ajuste basada en la transformada de Mellin se compara con las pruebas de Anderson-Darling y Kolmogorov-Smirnov.

Palabras Clave: Bondad de Ajuste, Estadística T^2 de Hotelling, Log-Cumulantes, Majorización, Estadísticas de Orden, Estimación de Parámetros.

Resumo

Nesta tese, fornecemos resultados em dois campos. No primeiro, a ideia principal é fazer comparações estocásticas utilizando a teoria da majoração. Investigamos condições sobre os parâmetros da classe Exponenciada Generalizada com distribuições-base iguais e diferentes. No segundo, para descrever os ajustes dos modelos aos dados por meio de medidas de bondade-de-ajuste, usamos a distribuição Weibull Inversa Transmutada e derivamos a Transformada Mellin desse modelo. Estimamos os parâmetros pelos métodos de Momentos, função de Máxima Verossimilhança e Log-Cumulantes. Uma análise interessante é realizada com a estatística T^2 de Hotelling. Nesta etapa, construímos diagramas de Log-Cumulantes e fornecemos elipses de confiança para os Log-Cumulantes. As aplicações nesta parte foram baseadas em conjuntos de dados de vida útil, tais como tempo de sobrevivência de componentes mecânicos, tempos de falha de filmes isolantes elétricos, e dados censurados de tempos de pacientes com câncer de bexiga. Apresentamos alguns resultados de simulação para verificar o desempenho dos métodos de momentos, função de Máxima Verossimilhança e Log-Cumulantes. Kolmogorov-Smirnov e Cramér-von Mises como critérios de bondade-de-ajuste são usados. Os métodos Monte Carlo, tais como bootstrap e Jackknife são usados para estimar as características da população. O desempenho do teste de bondade-de-ajuste baseado na transformada de Mellin é comparado com os testes Anderson-Darling e Kolmogorov-Smirnov.

Palavras Chave: Bondade de ajuste, estatística T^2 de Hotelling, Log-Cumulantes, Majoração, estatísticas de ordem, estimação de parâmetros.

Chapter 1

Introduction

Studies based on distribution functions have been applied in different areas of statistics and engineering, for example, survival analysis is one of that. This field use tools to study life times, survival times (Lee et al., 2007), failure time (Pescim et al., 2013) and reliability (Kececioglu, 2002; Meeker et al., 2021), among others. In the same way, new distribution functions have been emerging, and with them, the applications are trying to solve or improve some problems in the real life.

The motivation for studying stochastic comparisons and goodness-of-fit (GoF) theory was given in terms of distributions field. This as an alternative to expand the study of distributions in two areas of interest to researchers.

In the first part of this study, we are interested in comparing random variables, due to the fact, comparisons based on moments, for example, mean, variance and standard deviation, are not very informative, though they are simple to use. As a consequence, in the literature we can find how to compare variabilities of random variables based on their complete distribution functions (Müller and Stoyan, 2002; Balakrishnan et al., 2020).

On the other hand, using the Mellin Transform associated with a random variable has several advantages, among them simplifying calculations and providing new results in GoF measures. There are many GoF techniques, however for new distribution functions those techniques may sometimes not give good results of fits. To use them, depends on the methods that pretend to see how well a sample of the data with a given distribution describes its population (D'Agostino, 1986). The focus in that part of the document is based on the Mellin Transform and the T^2 statistic by Log-Cumulants.

This document contains independent chapters; for that reason, sequential reading is unnecessary. Now, we visualize the following objectives in this thesis.

Contributions to the science with higher advanced topics in applied statistics and from researchers' interest, specifically in stochastic comparisons, distribution functions and GoF measures are presented here.

Main Goals:

A) To analyze the basic majorization theory to compare random variables from the Exponentiated Generalized Class;

B) To propose a new measure of Goodness-of-Fit for the Transmuted Inverse Weibull distribution.

Specific Goals:

A1) To compare the maximum of independent random variables under certain conditions in the Exponentiated Generalized Class parameters;

A2) To provide new results in the Exponentiated Generalized Class context by using stochastic comparisons;

A3) To give a possible application of the stochastic comparisons.

B1) To build the Mellin Transform for Transmuted Inverse Weibull distribution;

B2) To derive the Log-Cumulants expressions of the Transmuted Inverse Weibull distribution;

B3) To propose a parameter estimation criteria based on Log-Cumulants from Transmuted Inverse Weibull distribution;

B4) To use the Mellin Transform to applied in a specific Goodness-of-Fit measure;

B5) To derive the T^2 statistic based on Log-Cumulants estimated;

B6) To verify the performance estimators by computational study with the Transmuted Inverse Weibull distribution.

B7) To compare the Moments, Maximum Likelihood and Log-Cumulants methods through the computational study.

B8) To compare the Kolmogorov Smirnov and Cramér-vom Mises statistics as Goodness-of-Fit criterias.

The thesis is directed with the aim to give new results in the statistic literature, specifically in the general field of distributions. To accomplish this, the work is organized as follows: In Chapter 2, some preliminary notions. Chapter 3, is about stochastic comparisons: The most straightforward way of comparing two distribution functions is by the comparing the associated means. However, such a comparison is based on only two single numbers (the means), and therefore it is often not very informative. In addition, the means sometimes do not exist (Shaked and Shanthikumar, 2007). One can find several forms of study on the underlying distributions in terms of their survival functions, failure rate, quantile, among others. These methods are more informative than those based on only numerical characteristics of distributions. Comparisons of random variables based on these functions often establish partial orders between them known as stochastic orders (Torrado and Lillo, 2013).

Here, we consider stochastic comparisons with the Exponentiated Generalized Class. Some relevant basic concepts on stochastic orderings, such as order statistics, Schur-convex, and Schur-concave concepts, are needed to understand the results. Ordering properties of lifetimes of parallel systems with equal and different distributions are presented. Examples and counterexamples illustrate our results. Finally, remarking that, for the moment, it is impossible to do a simulation study in this work because we need other tools that flee from the focus of our research. However, we give a possible application case.

Chapter 4, is about GoF measures through on the Mellin transform. The first tests hypotheses about whether a distribution fits well with the data. Anfinsen et al. (2011) mentioned that the procedure measures the discrepancy of the data sample concerning the distribution model. It provides a test statistic, which is used to decide whether the null hypothesis should be rejected or not rejected. Also, they spoke in that article about different types of GoF tests such as the well-known Chi-Square, Kolmogorov-Smirnov, empirical distribution function test, or the test based on regression. The second one, the Mellin Transform, is a simple transposition of the definitions of the characteristic functions from the Fourier field to the Mellin field, opening a new path and avoiding heavy calculations. In this sense, we propose an estimation method based on Log-Cumulants using the Mellin Transform. The parameter estimation is based on the Moments, Maximum Likelihood, and Log-Cumulants methods. And then, some applications in survival data are presented (Nicolas, 2006).

In Chapter 5, a simulation study is carried out. For different parameter scenarios and sample sizes, we find the average estimates for Transmuted Inverse Weibull model by using the methods of Moments, Maximum Likelihood estimation and Log-Cumulants. Also, we use the general GoF test approach, which uses distance statistics such as the Kolmogorov-Smirnov and Cramer-von Mises statistics and the power of the tests to estimate measures. Finally, in Chapter 6, the conclusions and future recommendations are given.

Chapter 2

Preliminary Notions

This chapter reviews some results needed to develop the following chapters. We present notions, concepts and definitions from the literature such as Balakrishnan and Torrado (2016) and Marshall et al. (2011), and they are necessary to understand and claryfy the content in each chapter. Next, two sections are presented. The first section is about general majorization theory concepts, and the second is about some Goodness-of-Fit theories. The notation here is the same used, specifically existing in the literature in each field.

2.1 Majorization Theory

Since 1979 the origins of majorization were in comparisons of inequalities in different manners from different fields, then different applications appeared in Marshall et al. (2011). The first comprehensive study of the stochastic comparisons and functions preserving the ordering of majorization was made by Schur. To refer to the concept of Schur-Convex functions, we must mention the partial ordering ($\preceq$) of any set $\mathbb{A} \subset \mathbb{R}^n$.

Definition 2.1.1. *(Order-Preserving) A real valued function ϕ defined on $\mathbb{R}$ which satisfy $\phi(\alpha_1) \leq \phi(\alpha_2)$ whenever $\alpha_1 \preceq \alpha_2$ with $\alpha_1, \alpha_2 \in \mathbb{A}$ are referred to as monotonic, isotonic, or order-preserving.*

Another concepts presented are parallel and series systems, which are presented below.

Definition 2.1.2. *(Series and Parallel System) A k-out-of-n system of n components functions if and only if at least k of the components function. The time to failure of a k-out-of-n system of n components with lifetimes $X_1, X_2, \ldots, X_n$, corresponds to the $(n-k+1)$-th order statistic, that is $X_{n-k+1:n}$. Thus the study of the lifetimes of k-out-of-n systems is equivalent to the study of the probability distributions of order statistics. A series system is an n-out-of-n system and a parallel system is a 1-out-of-n system. Thus the time to failure of* *a series system corresponds to the first order statistic while that of a parallel system corresponds to the largest order statistic.*

A series and parallel system illustration is giving in Figure A and Figure B, respectively.

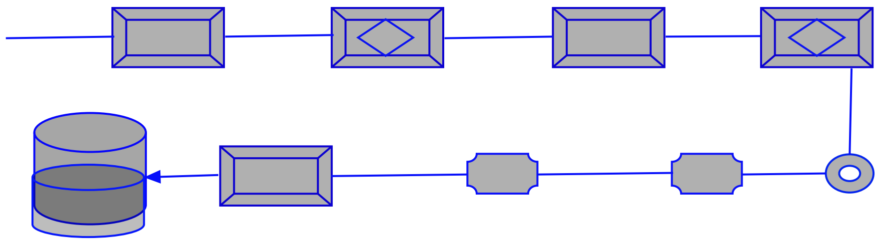


Figure A: Series System (own source)

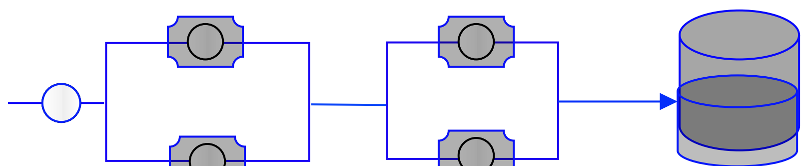


Figure B: Parallel system (own source)

Definition 2.1.3. *(Multiple-Outliers) A multiple-outlier model is a set of random variables $X_1, X_2, \ldots, X_n$ such that X_i is equal stochastically to X, denoted by $\left(X_i \overset{st}{=} X\right)$, for $i = 1, 2, \ldots, p$, and $X_i \overset{st}{=} Y$, for $i = p+1, p+2, \ldots, n$, where $1 \leq p < n$. It is further assumed that the two samples are independent.*

In Figure C, we illustrate the multiple-outlier concept in an applied Example 2.1.1.

Example 2.1.1. *Assume two subsystems connected, where each one is formed by n components. Without loss of generality, it consists of two repair lines for component parts of a machine. If the machine does not have the k-th damaged part, it goes to the next stage, even being able to go to the second line, until the machine comes out completely repaired.*

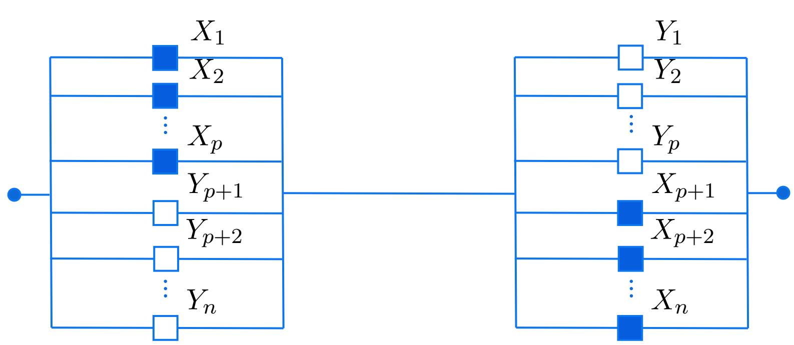


Figure C: Multiple-Outliers system illustration (own source)

2.2 Notions About Goodness-of-Fit

The fit of a distribution to the dataset can be verified through the GoF tests such as Kolmogorov-Smirnov and Cramér-Von Mises. These tests are based on the empirical distribution function and quantify a distance between the empirical distribution function of the sample and the cumulative distribution function of the reference distribution; for more details see Kececioglu (2002).

The Cramér-Von Mises test is defined by

$$W_n^2 = \frac{1}{12n} + \sum_{i=1}^{n} \left(F_0(x_{i:n}|\mathbf{\Theta}) - \frac{2i-1}{n} \right)^2 . \tag{2.1}$$

The Cramér-Von Mises estimators of the parameters $\mathbf{\Theta}$ are obtained by minimizing concerning $\mathbf{\Theta}$ the function (2.1).

The Kolmogorov-Smirnov test is the best-known and most widely used Goodness-of-Fit test based on the empirical distribution function due to the fact its relative simplicity and computational use. Is defined as

$$D_n = \underset{1\leq i\leq n}{max} \left\{ \frac{i}{n} - F(x_{i:n}|\mathbf{\Theta}), F(x_{i:n}|\mathbf{\Theta}) - \frac{i-1}{n} \right\} .$$

Where $F(\cdot)$ is the theoretical cumulative distribution function wholly specified, and $x_{i:n}$ are the order statistics of a statistical sample.

2.3 Bootstrap and Jackknife methods

Statistical resampling methods have been practical for parameter estimation, their fundamental idea is to estimate the characteristics of a statistic by obtaining repeated samples of the original data and making inferences about those resamplings. This section focuses on resample techniques such as Bootstrap and Jackknife.

i) **Bootstrap:** A first idea of a statistician is to study a sample (mean, median, variance) and then generalize the results to the population scientifically. Efron (1979) introduced a statistical method called bootstrap for setting standar errors, biases, confidence intervals, and others. Let us suppose that samples of the same size are repeatedly drawn from the population a large number of times to have a good approach to the distribution of a certain statistic, from the collection of its values arising from these repeated samples collecting information in an economical and timely manner.

   The idea behind bootstrap is to use the data of a sample study as a surrogate population to approximate the sampling distribution of a statistic, that is, to resample (with replacement) from the sample data at hand and create a large number of phantom samples known as bootstrap samples. The sample summary is then computed on each bootstrap samples (usually a few thousand). A histogram of the set of these computed values is referred to as the bootstrap distribution of the statistic (Singh and Xie, 2008).

ii) **Jackknife:** This method was proposed by M. H. Quenouille (1949). Later, John Tukey in 1956 refined this technique. Initially, this technique was to correct the bias, but later, Tukey used it to build confidence limits. This technique is similar to the bootstrap in that it involves resampling, but instead of sampling with replacement, the method samples without replacement (Miller, 1974).

Chapter 3

Stochastic Comparisons Of Parallel Systems With Exponentiated Generalized Class

Abstract

In this chapter, we study different stochastic comparisons for life times of parallel systems using the exponentiated generalized class. From equal and different baseline distribution, we compare the maximum of two independent samples under certain conditions in the parameters, using the reversed hazard rate order and the usual stochastic order. Finally, we show examples and counterexamples.

Keywords: Majorization Theory, Order Statistics, Parallel System, Reliability.

3.1 Introduction

Results on stochastic comparisons using order statistics and majorization theory have been obtained in recent years. Some of these studies showed comparisons between random variables following a certain distribution. It may refer for example to Dykstra et al. (1997), Kochar (2012), Torrado and Lillo (2013), Torrado and Kochar (2015), Torrado (2015), Kundu and Chowdhury (2016), Torrado (2017), Biswas and Gupta (2019), Balakrishnan et al. (2020), among many others. Recently, Kundu and Chowdhury (2018), Chowdhury (2019), Kayal (2019) and Kayal and Nanda (2020) used generalized families for comparing parallel systems, that is, they use the maximum statistic for comparisons.

Between recent publications, Kayal and Kundu (2021), considered stochastic comparison between two systems when the associated systems' components do not receive random shocks and also they derived majorization-type partial order based sufficient conditions for the comparisons of the extreme order statistics in the sense of various stochastic orders such as the usual stochastic, likelihood ratio and hazard rate orders. Kayal and Kundu (2021), considered stochastic comparison of the largest and the smallest order statistics arising from heterogeneous log-logistic distributions. Kayal et al. (2022), analyzed stochastic comparison of two finite mixture models with respect to usual stochastic order when the mixture components have a general family of distributions. Sattari et al. (2022), examined the problem of stochastic comparisons of series and parallel systems with independent heterogeneous new extended Weibull distributed components. Barmalzan et al. (2022), analyzed stochastic comparisons of parallel and series systems consisting of dependent multiple-outlier scale components with Archimedean copula as the joint distribution in terms of likelihood ratio and dispersive orders.

Following the line of generalized models, Gupta et al. (1998), were the first to propose a generalization of the standard exponential distribution, known as Exponential Exponentiated (EE) distribution. Let X be a random variable following an EE distribution with parameters $\lambda > 0$, $\alpha > 0$, the scale and shape parameters, respectively, denoted by $X \sim \text{EE}(\alpha, \lambda)$, then its cumulative distribution function (CDF) is given by

$$F(x) = \left(1 - \mathrm{e}^{-\lambda x}\right)^{\alpha} I_{(0,\infty)}(x),$$

where $I_A(\cdot)$ is the indicator function that defines the distribution support.

To further extend the results, Cordeiro et al. (2013), proposed a class of distributions called Exponentialized Generalized (EG) class. Then, if $X \sim \text{EG}(\alpha, \beta, G(\cdot))$ its cdf is defined by

$$F(x) = \left[1 - (1 - G(x))^{\alpha}\right]^{\beta}, \tag{3.1}$$

where $\alpha > 0$ and $\beta > 0$ are additional shape parameters and $G(\cdot)$ is a continuous cdf. Thus, the EG class allows greater flexibility and control over its tails providing better fits. Observe that, now the support is not necessary positive real numbers. The probability density function (pdf) of the EG class, where $g(\cdot)$ is the density function associated to the baseline distribution $G(\cdot)$, is given by

$$f(x) = \alpha\beta g(x)\,(1 - G(x))^{\alpha-1}\left[1 - (1 - G(x))^{\alpha}\right]^{\beta-1}. \tag{3.2}$$

Now, we want to highlight an interesting observation, if we take $\alpha = 1$ in (3.1), or $\beta = 1$, we obtain the Lehmann type I and II, respectively (Lehmann, 1953). The aspects in the paper of Lehmann was very important to developed the distributions literature, simply with different choices of baseline distribution. Balakrishnan (2021) mentioned that all these distributions, are part of the Lehmann family, each one with their properties in inferential aspects and used for modeling purposes.

From (3.1) and (3.2), we obtain the reversed hazard (failure) rate function as follows

$$\tilde{r}(x) = \frac{\alpha\beta r_G(x)\,(1 - G(x))^{\alpha}}{1 - (1 - G(x))^{\alpha}}, \tag{3.3}$$

where $r_G(x)$ is the hazard rate function associated to $G(\cdot)$ distribution.

The goal of this chapter is to compare the maximum of two independent samples, from the same and different baseline distributions, under certain conditions in the parameters of the exponentiated generalized class. The examples/counterexamples were based on the distributions EG Fréchet (EGF) and EG normal (EGN) by Cordeiro et al. (2013), EG Birnbaum-Saunders (EGBS) by Cordeiro and Lemonte (2014), EG Gumbel (EGGu) by Andrade et al. (2015), EG Nadarajah-Haghighi (EGNH) by VedoVatto et al. (2016) and EG Kumaraswamy (EGKw) by Elgarhy et al. (2018).

If the random variables $X_1, X_2, \ldots, X_n$ are arranged in ascending order of magnitude, then the i-th smallest of $X_i's$ is denoted by $X_{i:n}$ and those ordered quantities $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$, are called order statistics in which $X_{i:n}$ is the i-th order statistic. The smallest and largest of these statistics, known as minimum and maximum, play a fundamental role in statistics, reliability theory, applied probability, among other fields. Thus, the $(n-k+1)$-th order statistics in a sample of size n represents the life length of a system k-out-of-n, that consists of n components with independent and identically distributed life lengths. Here, all n components start working simultaneously, and the system works, if at least k components function; i.e., the system fails if $(n-k+1)$ or more components fail. Special systems cases are series and parallel systems, studied in Fang and Zhang (2013), Kundu et al. (2014), Torrado and Kochar (2015) and Torrado (2017), among others.

This research is organized as follow: In section 3.2 some basic notions about majorization theory, important lemmas and theorems are presented do develop our research. Section 3.3 the main result, we investigate conditions on the parameters of the EG class to do stochastic comparisons and finally we present the conclusions.

3.2 Notations, definitions and some preliminary results

In this section, we introduce some definitions and concepts that are fundamental to understand stochastic inequalities. For more information on this theory, see Marshall et al. (2011) and Shaked and Shanthikumar (2007).

Definition 3.2.1. *Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)$ be vectors in $\mathbb{R}^n$. Then,*

(i) $\boldsymbol{\alpha}$ is said to majorize $\boldsymbol{\beta}$ ($\boldsymbol{\alpha} \overset{m}{\succeq} \boldsymbol{\beta}$) if

$$\sum_{i=1}^{j} \alpha_{i:n} \leq \sum_{i=1}^{j} \beta_{i:n}, \textit{ for all } j = 1, 2 \ldots, n-1 \textit{ and } \sum_{i=1}^{n} \alpha_{i:n} = \sum_{i=1}^{n} \beta_{i:n},$$

(ii) $\boldsymbol{\alpha}$ *is said to weakly supermajorize* $\boldsymbol{\beta}$ ($\boldsymbol{\alpha} \overset{w}{\succeq} \boldsymbol{\beta}$) *if*

$$\sum_{i=1}^{j} \alpha_{i:n} \leq \sum_{i=1}^{j} \beta_{i:n}, \textit{ for all } j = 1, 2 \ldots, n,$$

(iii) $\boldsymbol{\alpha}$ *is said to weakly submajorize* $\boldsymbol{\beta}$ ($\boldsymbol{\alpha} \succeq_w \boldsymbol{\beta}$) *if*

$$\sum_{i=j}^{n} \alpha_{i:n} \geq \sum_{i=j}^{n} \beta_{i:n}, \textit{ for all } j = 1, 2 \ldots, n.$$

Remark 3.2.1. *Note that* $\boldsymbol{\alpha} \overset{m}{\succeq} \boldsymbol{\beta}$ *implies* $\boldsymbol{\alpha} \overset{w}{\succeq} \boldsymbol{\beta}$ *and* $\boldsymbol{\alpha} \succeq_w \boldsymbol{\beta}$.

Let X and Y two absolutely continuous random variables with cumulative distribution functions $F(\cdot)$ and $H(\cdot)$, probability density functions $f(\cdot)$ and $h(\cdot)$, survival functions $\bar{F}(\cdot) = 1 - F(\cdot)$ and $\bar{H}(\cdot) = 1 - H(\cdot)$, hazart rate functions $r = f(\cdot)/\bar{F}(\cdot)$ and $s = h(\cdot)/\bar{H}(\cdot)$, reversed hazard rate $\tilde{r} = f(\cdot)/F(\cdot)$ and $\tilde{s} = h(\cdot)/H(\cdot)$, respectively.

Definition 3.2.2. *X is said to be smaller than Y in*

1. *likelihood ratio (denoted as $X \leq_{lr} Y$) if $h(x)/f(x)$ is increasing in $x \geq 0$;*
2. *hazard rate order (denoted as $X \leq_{hr} Y$) if $r(x) \geq s(x)$, or equivalently $\bar{H}(x)/\bar{F}(x)$ is increasin in x, for all $x \geq 0$;*
3. *reversed hazard rate order (denoted as $X \leq_{rhr} Y$) if $\tilde{r}(x) \leq \tilde{s}(x)$, or equivalently $H(x)/F(x)$ is increasin in x for all $x \geq 0$;*
4. *usual stochastic order (denoted as $X \leq_{st} Y$) if $F(x) \geq H(x)$, or equivalently $\bar{F}(x) \leq \bar{H}(x)$ for all x.*

Remark 3.2.2. *Note that,* $X \leq_{lr} Y \Longrightarrow X \leq_{rhr} Y (X \leq_{hr} Y) \Longrightarrow X \leq_{st} Y$.

For more datails see, Marshall et al. (2011), Chapter 17 Stochastic Ordering, Section A: Some Basic Stochastic Orders, A.1. Definition (st), A.7. Definition. (hr), A.8., A.11. Definition (lr).

Now, let us to introduce some notations:

(i) $\boldsymbol{D} = \{(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n : \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n\}$.

(ii) $\boldsymbol{D}_+ = \{(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n : \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n > 0\}$.

(iii) $\boldsymbol{E} = \{(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n : \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n\}$.

(iv) $\boldsymbol{E}_+ = \{(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n : 0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n\}$.

The following definition and lemmas are well known and can be found in (Marshall et al., 2011) and (Kundu et al., 2016).

Definition 3.2.3. *Let I^n be an $n-$dimensional Euclidean space where $I^n \subset \mathbb{R}^n$. The function $\varphi : I^n \longrightarrow \mathbb{R}$ is Schur-convex (resp. Schur-concave) on I^n if $\boldsymbol{\alpha} \overset{m}{\succeq} \boldsymbol{\beta}$ implies $\varphi(\boldsymbol{\alpha}) \geq$ (resp. $\leq$) $\varphi(\boldsymbol{\beta})$ for all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in I^n$.*

Lemma 3.2.1. *(Theorem 3.1 of Kundu et al. (2016)). Let $I \subseteq \mathbb{R}$ an interval. Consider $\kappa : I \longrightarrow \mathbb{R}$, a continuous function. Let $\varphi(\boldsymbol{x}) = \sum_{i=1}^{n} u_i \kappa(x_i)$ with $\boldsymbol{x} \in \boldsymbol{D}$.*

(a) If $\boldsymbol{u}= (u_1, u_2, \ldots, u_n) \in \boldsymbol{D}_+$ and

(i) κ is increasing and convex, then $\varphi(\boldsymbol{x})$ is Schur-convex on $\boldsymbol{D}$;

(ii) κ is decreasing and concave, then $\varphi(\boldsymbol{x})$ is Schur-concave on $\boldsymbol{D}$.

(b) If $\boldsymbol{u}= (u_1, u_2, \ldots, u_n) \in \boldsymbol{E}_+$ and

(i) κ is increasing and concave, then $\varphi(\boldsymbol{x})$ is Schur-concave on $\boldsymbol{D}$;

(ii) κ is decreasing and convex, then $\varphi(\boldsymbol{x})$ is Schur-convex on $\boldsymbol{D}$.

Lemma 3.2.2. *(Theorem 3.2 of Kundu et al. (2016)). Let $I \subseteq \mathbb{R}$ an interval. Let $\kappa : I \longrightarrow \mathbb{R}$ be a continuous function. Let $\varphi(\boldsymbol{x}) = \sum_{i=1}^{n} u_i \kappa(x_i)$ with $\boldsymbol{x} \in \boldsymbol{E}$.*

(a) If $\boldsymbol{u}= (u_1, u_2, \ldots, u_n) \in \boldsymbol{D}_+$ and

(i) κ is increasing and concave, then $\varphi(\boldsymbol{x})$ is Schur-concave on $\boldsymbol{E}$;

(ii) κ is decreasing and convex, then $\varphi(\boldsymbol{x})$ is Schur-convex on $\boldsymbol{E}$.

(b) If $\boldsymbol{u} = (u_1, u_2, \ldots, u_n) \in \boldsymbol{E}_+$ and

(i) κ is increasing and convex, then $\varphi(\boldsymbol{x})$ and Schur-convex on $\boldsymbol{E}$;

(ii) κ is decreasing and concave, then $\varphi(\boldsymbol{x})$ and Schur-concave on $\boldsymbol{E}$.

Lemma 3.2.3. *(Marshall et al. (2011)). Let I^n be an $n-$dimensional Euclidean space where $I^n \subset \mathbb{R}^n$. Let $\varphi : I^n \longrightarrow \mathbb{R}$. Then*

(i) $(\alpha_1, \alpha_2, \ldots, \alpha_n) \succeq_w (\beta_1, \beta_2, \ldots, \beta_n)$ *implies* $\varphi(\alpha_1, \alpha_2, \ldots, \alpha_n) \geq$ *(resp. $\leq$)* $\varphi(\beta_1, \beta_2, \ldots, \beta_n)$ *if, and only if, φ is increasing (resp. decreasing) and Schur-convex (resp. Schur-concave) on I^n.*

(ii) $(\alpha_1, \alpha_2, \ldots, \alpha_n) \overset{w}{\succeq} (\beta_1, \beta_2, \ldots, \beta_n)$ *implies* $\varphi(\alpha_1, \alpha_2, \ldots, \alpha_n) \geq$ *(resp. $\leq$)* $\varphi(\beta_1, \beta_2, \ldots, \beta_n)$ *if, and only if, φ is decreasing (resp. increasing) and Schur-convex (resp. Schur-concave) on I^n.*

Next, we present a Lemma which will be useful to prove some results in the following sections. The proof follows similar arguments taken from Laniado and Lillo (2014).

Lemma 3.2.4. *Let $\bar{G}(x) = 1 - G(x)$. The function $k(\theta) = \dfrac{\theta}{(\bar{G}(x))^{-\theta} - 1}$ is decreasing and convex in $\theta > 0$.*

Proof. First, we need to derive $k(\theta)$ with respect to θ to show that it's decreasing. Let $\bar{G}(x) = 1 - G(x)$.

$$\begin{aligned}
k'(\theta) &= \frac{(\bar{G}(x))^{-\theta} - 1 - \theta\left(-(\bar{G}(x))^{-\theta}\log(\bar{G}(x))\right)}{\left((\bar{G}(x))^{-\theta} - 1\right)^2} \\
&= \frac{(\bar{G}(x))^{2\theta}\left[(\bar{G}(x))^{-\theta} - 1 + \theta\left((\bar{G}(x))^{-\theta}\log(\bar{G}(x))\right)\right]}{(\bar{G}(x))^{2\theta}\left[\left((\bar{G}(x))^{-\theta} - 1\right)^2\right]} \\
&= \frac{(\bar{G}(x))^{\theta}\overbrace{\left(1 + \theta\log(\bar{G}(x)) - (\bar{G}(x))^{\theta}\right)}^{k_1(\theta)}}{\left((\bar{G}(x))^{\theta} - 1\right)^2}.
\end{aligned}$$

For $k'(\theta)$ to be negative, we will show that $k_1(\theta) \leq 0$. Observe that, if $\theta = 0$ then $k_1(\theta) = 0$.

$$\begin{aligned}
k_1'(\theta) &= \log(\bar{G}(x)) - (\bar{G}(x))^{\theta}\log(\bar{G}(x)) \\
&= \log(\bar{G}(x))\left(1 - (\bar{G}(x))^{\theta}\right),
\end{aligned}$$

as $(\bar{G}(x)) \in (0,1)$, $\log(\bar{G}(x))$ it's not positive. Implies that $k(\theta)$ is decreasing.

Let us to show the convexity of $k(\theta)$:

$$
\begin{aligned}
k''(\theta) \;=\;& \Big[(\bar{G}(x))^\theta \log\big(\bar{G}(x)\big)(1-(\bar{G}(x))^\theta+\theta\log\big(\bar{G}(x)\big))+(\bar{G}(x))^\theta(-(\bar{G}(x))^\theta\log\big(\bar{G}(x)\big)+\log\big(\bar{G}(x)\big))\Big]\Big[(\bar{G}(x))^\theta-1\Big]^2 \\
& -\Big[2((\bar{G}(x))^\theta-1)(\bar{G}(x))^\theta\log\big(\bar{G}(x)\big)\Big]\Big[(\bar{G}(x))^\theta(1+\theta\log\big(\bar{G}(x)\big)-(\bar{G}(x))^\theta)\Big] \\
& /\Big((\bar{G}(x))^\theta-1\Big)^4 \\
=\;& \cancel{\Big((\bar{G}(x))^\theta-1\Big)}\Big\{\Big[(\bar{G}(x))^\theta \log\big(\bar{G}(x)\big)(1-(\bar{G}(x))^\theta+\theta\log\big(\bar{G}(x)\big))+(\bar{G}(x))^\theta(-(\bar{G}(x))^\theta\log\big(\bar{G}(x)\big)+\log\big(\bar{G}(x)\big))\Big]\Big[(\bar{G}(x))^\theta-1\Big] \\
& -\Big[2(\bar{G}(x))^\theta\log\big(\bar{G}(x)\big)\Big]\Big[(\bar{G}(x))^\theta(1+\theta\log\big(\bar{G}(x)\big)-(\bar{G}(x))^\theta)\Big]\Big\} \\
& /\Big((\bar{G}(x))^\theta-1\Big)^{\cancel{4}3} \\
=\;& \frac{(\bar{G}(x))^\theta\log\big(\bar{G}(x)\big)(-\theta(\bar{G}(x))^\theta\log\big(\bar{G}(x)\big)+2(\bar{G}(x))^\theta-2-\theta\log\big(\bar{G}(x)\big))}{((\bar{G}(x))^\theta-1)^3} \\
=\;& -\frac{(\bar{G}(x))^\theta\log\big(\bar{G}(x)\big)\overbrace{\Big[\theta\log\big(\bar{G}(x)\big)\Big(1+(\bar{G}(x))^\theta\Big)+2\Big(1-(\bar{G}(x))^\theta\Big)\Big]}^{l_1(\theta)}}{((\bar{G}(x))^\theta-1)^3}.
\end{aligned}
$$

We need to show that $k''(\theta) > 0$. As $\log\big(\bar{G}(x)\big)$ and $\big((\bar{G}(x))^\theta-1\big)^3$ are not positives, we have to show that $l_1(\theta) \leq 0$.

$$
\begin{aligned}
l_1'(\theta) &= \log\big(\bar{G}(x)\big)\Big(1+(\bar{G}(x))^\theta\Big)+\Big((\bar{G}(x))^\theta\log\big(\bar{G}(x)\big)\Big)\left(\theta\log\big(\bar{G}(x)\big)\right)+2\Big(-(\bar{G}(x))^\theta\log\big(\bar{G}(x)\big)\Big) \\
&= \log\big(\bar{G}(x)\big)+(\bar{G}(x))^\theta\log\big(\bar{G}(x)\big)+\theta(\bar{G}(x))^\theta\left(\log\big(\bar{G}(x)\big)\right)^2-2(\bar{G}(x))^\theta\log\big(\bar{G}(x)\big) \\
&= \log\big(\bar{G}(x)\big)-(\bar{G}(x))^\theta\log\big(\bar{G}(x)\big)+\theta(\bar{G}(x))^\theta\left(\log\big(\bar{G}(x)\big)\right)^2 \\
&= \log\big(\bar{G}(x)\big)\underbrace{\Big(1-(\bar{G}(x))^\theta+\theta(\bar{G}(x))^\theta\log\big(\bar{G}(x)\big)\Big)}_{l_2(\theta)}.
\end{aligned}
$$

Given that $\theta > 0$, $l_2(\theta)$ it's not increasing in $\bar{G}(x) \in (0,1)$ because

$$
\begin{aligned}
\frac{\partial}{\partial\bar{G}(x)}l_2(\theta) &= \theta\Big(\theta(\bar{G}(x))^{\theta-1}\log\big(\bar{G}(x)\big)+(\bar{G}(x))^{-1}(\bar{G}(x))^\theta\Big)-\theta(\bar{G}(x))^{\theta-1} \\
&= \theta(\bar{G}(x))^{\theta-1}\left(\theta\log\big(\bar{G}(x)\big)+1\right)-\theta(\bar{G}(x))^{\theta-1} \\
&= \theta^2(\bar{G}(x))^{\theta-1}\log\big(\bar{G}(x)\big) \leq 0.
\end{aligned}
$$

Therefore, $l_2(\theta) \geq 0$. According to this, $l_1'(\theta) \leq 0$, this is $l_1(\theta)$ it's not increasing in $\theta > 0$, implies $l_1(\theta) \leq 0$. Here, we obtain the convexity of $k(\theta)$.

$\square$

3.3 Main Results

In this section, we investigate conditions on the parameters of the EG class with equal and different baseline distributions in the stochastic comparisons.

Let $X_i \sim \text{EG}(\alpha_i, \beta_i, G(\cdot))$ and $Y_i \sim EG(\gamma_i, \delta_i, G(\cdot))$, with $i = 1, 2, \ldots, n$, two sets of independent random variables in which the baseline distribution $G(\cdot)$ is continuous and common to both sets of random variables.

Let us denote $F_{n:n}(\cdot)$ and $H_{n:n}(\cdot)$ the cdfs of $X_{n:n}$ and $Y_{n:n}$, respectively.

Therefore

$$F_{n:n}(x) = \prod_{i=1}^{n} \left(1 - [1 - G(x)]^{\alpha_i}\right)^{\beta_i},$$

and

$$H_{n:n}(x) = \prod_{i=1}^{n} \left(1 - [1 - G(x)]^{\gamma_i}\right)^{\delta_i}.$$

Then, the pdfs are

$$f_{n:n}(x) = \prod_{i=1}^{n} F_i(x) \left(\sum_{i=1}^{n} \tilde{r}_i(x) \right),$$

and

$$h_{n:n}(x) = \prod_{i=1}^{n} H_i(x) \left(\sum_{i=1}^{n} \tilde{s}_i(x) \right),$$

where $\tilde{r}_i(x)$ and $\tilde{s}_i(x)$ are defined in (3.3) for $i = 1, 2, \ldots, n$.

Now, the reversed failure rate functions of $X_{n:n}$ and $Y_{n:n}$, are

$$\tilde{r}_{n:n}(x) = r_G(x) \sum_{i=1}^{n} \frac{\alpha_i}{(1 - G(x))^{-\alpha_i} - 1} \beta_i, \tag{3.4}$$

and

$$\tilde{s}_{n:n}(x) = r_G(x) \sum_{i=1}^{n} \frac{\gamma_i}{(1 - G(x))^{-\gamma_i} - 1} \delta_i, \tag{3.5}$$

respectively, where r_G is the reversed failure rate function associated to the baseline distribution function $G(\cdot)$.

Let I^n be an $n-$dimensional Euclidean space. Consider the vectors $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)$, $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_n)$, $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_n)$ in I^n.

Theorem 3.3.1. *Let X_i and Y_i be two sets of mutually independent random variables with $X_i \sim EG(\alpha_i, \beta_i, G(\cdot))$ and $Y_i \sim EG(\alpha_i, \delta_i, G(\cdot))$, $i = 1, 2, \ldots, n$.*

(i) If $\boldsymbol{\beta} \succeq_w \boldsymbol{\delta}$*,* $\boldsymbol{\alpha} \in \boldsymbol{E}_+$*,* $\boldsymbol{\beta}, \boldsymbol{\delta} \in \boldsymbol{D}$ *or* $\boldsymbol{\alpha} \in \boldsymbol{D}_+$*,* $\boldsymbol{\beta}, \boldsymbol{\delta} \in \boldsymbol{E}$ *then* $X_{n:n} \geq_{rhr} Y_{n:n}$*;*

(ii) If $\boldsymbol{\beta} \overset{w}{\succeq} \boldsymbol{\delta}$*,* $\boldsymbol{\alpha} \in \boldsymbol{D}_+$*,* $\boldsymbol{\beta}, \boldsymbol{\delta} \in \boldsymbol{D}$ *or* $\boldsymbol{\alpha} \in \boldsymbol{E}_+$*,* $\boldsymbol{\beta}, \boldsymbol{\delta} \in \boldsymbol{E}$ *then* $X_{n:n} \leq_{rhr} Y_{n:n}$*.*

Proof. From (3.4) and (3.5), $X_{n:n} \geq_{rhr} (\leq_{rhr}) Y_{n:n}$ if

$$\sum_{i=1}^{n} \frac{\alpha_i}{\left(\bar{G}(x)\right)^{-\alpha_i} - 1}\beta_i \geq (\leq) \sum_{i=1}^{n} \frac{\alpha_i}{\left(\bar{G}(x)\right)^{-\alpha_i} - 1}\delta_i.$$

Let $\varphi(\boldsymbol{\beta}) = \sum_{i=1}^{n} u_i \kappa(\beta_i)$, where $\kappa(\beta_i) = \beta_i$ and $u_i = \frac{\alpha_i}{\left(\bar{G}(x)\right)^{-\alpha_i} - 1}$. Then, we need to study the Schur-convexity (Schur-concavity) of φ.

It is evident that κ is increasing, convex and concave. From Lemma 3.2.4, we know that u_i is decreasing in α_i, then $\boldsymbol{\alpha} \in \boldsymbol{E}_+(\boldsymbol{D}_+)$ implies $\boldsymbol{u} \in \boldsymbol{D}_+(\boldsymbol{E}_+)$.

Actually, if $\boldsymbol{\alpha} \in \boldsymbol{E}_+(\boldsymbol{D}_+)$, $\boldsymbol{\beta}, \boldsymbol{\delta} \in \boldsymbol{D}(\boldsymbol{E})$ and $\boldsymbol{\beta} \overset{m}{\succeq} \boldsymbol{\delta}$ then $X_{n:n} \geq_{rhr} Y_{n:n}$. Evidently, $\varphi(\boldsymbol{\beta})$ is increasing in β_i. Now, from Lemma 3.2.2 (b.i) and Lemma 3.2.1 (a.i), we have that φ is Schur-convex, i.e., $X_{n:n} \geq_{rhr} Y_{n:n}$.

Analogously, if $\boldsymbol{\alpha} \in \boldsymbol{D}_+(\boldsymbol{E}_+)$, $\boldsymbol{\beta}, \boldsymbol{\delta} \in \boldsymbol{D}(\boldsymbol{E})$ and $\boldsymbol{\beta} \overset{m}{\succeq} \boldsymbol{\delta}$ then $X_{n:n} \leq_{rhr} Y_{n:n}$. Clearly, $\varphi(\boldsymbol{\beta})$ is increasing in β_i, from Lemma 3.2.2 (a.i) and Lemma 3.2.1 (b.i), we obtain that φ is Schur-concave, i.e., $X_{n:n} \leq_{rhr} Y_{n:n}$. □

Table 3.1 displays the cdf, pdf, parameters and supports of baseline distributions of EG class used in the study, examples and counterexamples.

The next counterexample shows that the result of Theorem 3.3.1 cannot be extended up to lr ordering.

Counterexample 3.3.1. *Consider* $x > 0$*. Let* $X_i \sim$*EGBS*$(\alpha_i, \beta_i, G(\cdot))$ *and* $Y_i \sim$*EGBS*$(\alpha_i, \delta_i, G(\cdot))$*. If* $\boldsymbol{\alpha} \in \boldsymbol{D}_+$*,* $\boldsymbol{\beta}, \boldsymbol{\delta} \in \boldsymbol{D}$*, with* $\boldsymbol{\alpha} = (8, 3.1, 1.3)$*,* $\boldsymbol{\beta} = (4.3, 3.2, 2)$ *and* $\boldsymbol{\delta} = (4.4, 3.1, 2)$*, we have in Figure 3.1* $X_{3:3} \not\leq_{lr} Y_{3:3}$*. Notice that even preserving the majorization* $\boldsymbol{\beta} \overset{m}{\succeq} \boldsymbol{\delta}$*, is not possible to extend the result to likelihood rate order.*

EG	CDF	PDF	PARAMETERS	SUPPORT
EGF	$F(x)=\left[1-\left(1-\mathrm{e}^{-(\frac{\sigma}{x})^{\lambda}}\right)^{\alpha}\right]^{\beta}$	$f(x)=\alpha\beta\lambda\sigma^{\lambda}x^{-(\lambda+1)}\,\mathrm{e}^{-(\frac{\sigma}{x})^{\lambda}}\left(1-\mathrm{e}^{-(\frac{\sigma}{x})^{\lambda}}\right)^{\alpha-1}\left[1-\left(1-\mathrm{e}^{-(\frac{\sigma}{x})^{\lambda}}\right)^{\alpha}\right]^{\beta-1}$	$\alpha>0,\beta>0,\lambda>0,\sigma>0$	$I_{(0,\infty)}(x)$
EGN	$F(x)=\left[1-\left(1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right)^{\alpha}\right]^{\beta}$	$f(x)=\alpha\beta\sigma^{-1}\left(1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right)^{\alpha-1}\left[1-\left(1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right)^{\alpha}\right]^{\beta-1}\phi\left(\frac{x-\mu}{\sigma}\right)$	$\alpha>0,\beta>0,\mu\in\mathbf{R},\sigma>0$	$I_{(-\infty,\infty)}(x)$
EGBS	$F(x)=[1-\Phi(-v_x)^{\alpha}]^{\beta}$	$f(x)=\alpha\beta V_x\phi(v_x)\Phi(-v_x)^{\alpha-1}\left[1-\Phi(-v_x)^{\alpha}\right]^{\beta-1}$	$\alpha>0,\beta>0,v_x=\alpha^{-1}\rho(x/\beta)$	$I_{(0,\infty)}(x)$
EGGu	$F(x)=\left[1-\left(1-\mathrm{e}^{-\mathrm{e}^{-\frac{x-\mu}{\sigma}}}\right)^{\alpha}\right]^{\beta}$	$f(x)=\alpha\beta\sigma^{-1}\,\mathrm{e}^{-\frac{x-\mu}{\sigma}}\,\mathrm{e}^{-\mathrm{e}^{-\frac{x-\mu}{\sigma}}}\left(1-\mathrm{e}^{-\mathrm{e}^{-\frac{x-\mu}{\sigma}}}\right)^{\alpha-1}\left[1-\left(1-\mathrm{e}^{-\mathrm{e}^{-\frac{x-\mu}{\sigma}}}\right)^{\alpha}\right]^{\beta-1}$	$\alpha>0,\beta>0,\mu\in\mathbf{R},\sigma>0$	$I_{(-\infty,\infty)}(x)$
EGNH	$F(x)=\left[1-\left(\mathrm{e}^{1-(1+ax)^{b}}\right)^{\alpha}\right]^{\beta}$	$f(x)=\frac{ab\alpha\beta(1+ax)^{b-1}\left(\mathrm{e}^{1-(1+ax)^{b}}\right)^{\alpha}}{\left[1-\left(\mathrm{e}^{1-(1+ax)^{b}}\right)^{\alpha}\right]^{1-\beta}}$	$\alpha>0,\beta>0,a>0,1\neq b>0$	$I_{(0,\infty)}(x)$
EGKw	$F(x)=[1-(1-x^{a})^{b\alpha}]^{\beta}$	$f(x)=ab\alpha\beta x^{a-1}(1-x^{a})^{b\alpha-1}[1-(1-x^{a})^{b\alpha}]^{\beta-1}$	$\alpha>0,\beta>0,a>0,b>0$	$I_{(0,1)}(x)$

Table 3.1: EG class distributions used in the examples and counterexamples

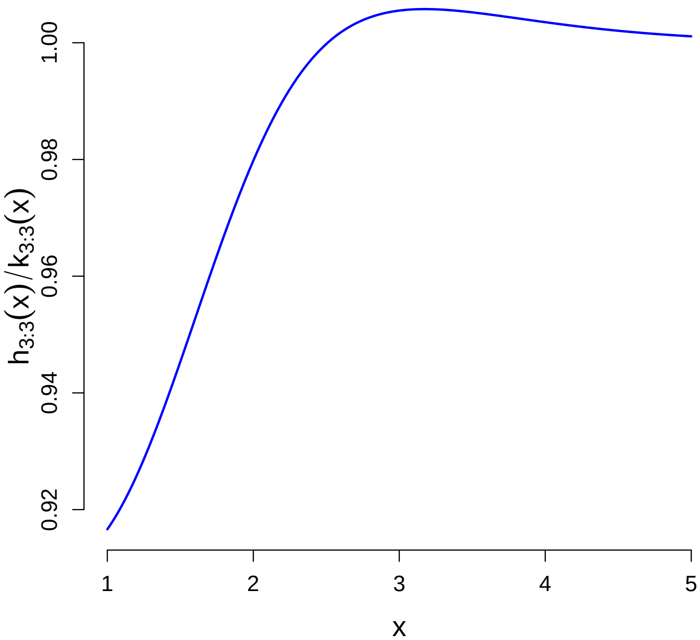


Figure 3.1: Plot of $h_{3:3}/f_{3:3}$, with $\boldsymbol{\alpha} \in \boldsymbol{D}_{+}, \boldsymbol{\beta}, \boldsymbol{\delta} \in \boldsymbol{D}$

The next example ilustrate the result of Theorem 3.3.1 when $\boldsymbol{\beta} \overset{w}{\succeq} \boldsymbol{\delta}$.

Example 3.3.1. *Let $x \in \mathbb{R}$. Let X_i and Y_i following EG class. Let $X_i \sim$EGGu$(\alpha_i, \beta_i, G(\cdot))$ and $Y_i \sim$EGGu$(\alpha_i, \delta_i, G(\cdot))$ with $\boldsymbol{\alpha} = (9, 8, 3) \in \boldsymbol{D}_{+}$, $\boldsymbol{\beta} = (4.3, 3.1, 2) \in \boldsymbol{D}$ and $\boldsymbol{\delta} = (4.3, 3.2, 2.2) \in \boldsymbol{D}$. We can observe that $\boldsymbol{\beta} \overset{w}{\succeq} \boldsymbol{\delta}$. Hence, $X_{n:n} \leq_{rhr} Y_{n:n}$ as evident from Figure 3.2.*

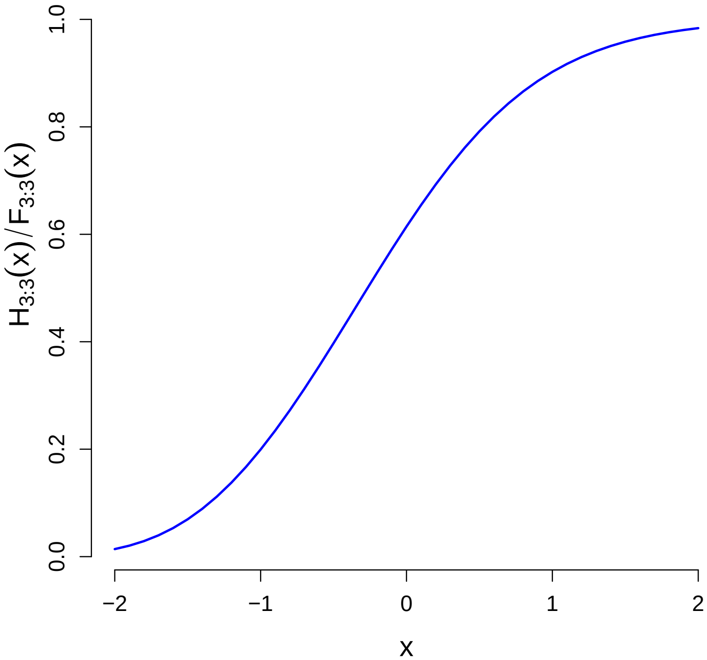


Figure 3.2: Plot of $H_{3:3}/F_{3:3}$, with $\boldsymbol{\alpha} \in \boldsymbol{D}_+, \boldsymbol{\beta}, \boldsymbol{\delta} \in \boldsymbol{D}$

Now we can ask, what will happen if $\boldsymbol{\alpha}$ supermajorizes $\boldsymbol{\gamma}$ while the parameters $\boldsymbol{\beta}$ and $\boldsymbol{\delta}$ are equal? The following theorem answers this question.

Theorem 3.3.2. *Let $X_i \sim EG(\alpha_i, \beta_i, G(\cdot))$ and $Y_i \sim EG(\gamma_i, \beta_i, G(\cdot))$ two sets of mutually independent random variables, with $i = 1, 2, \ldots, n$ and $G(\cdot)$ common to both. If $\boldsymbol{\alpha} \overset{w}{\succeq} \gamma$, and $\boldsymbol{\alpha}, \boldsymbol{\gamma} \in \boldsymbol{D}$, $\boldsymbol{\beta} \in \boldsymbol{E}_+$ or $\boldsymbol{\alpha}, \boldsymbol{\gamma} \in \boldsymbol{E}$, $\boldsymbol{\beta} \in \boldsymbol{D}_+$, then $X_{n:n} \geq_{rhr} Y_{n:n}$.*

Proof. From (3.4) and (3.5), we know that $\tilde{r}_{n:n}(x) \geq \tilde{s}_{n:n}(x)$ if, and only if $\varphi(\boldsymbol{\alpha}) = \sum_{i=1}^{n} \beta_i \kappa(\alpha_i) \geq \sum_{i=1}^{n} \beta_i \kappa(\gamma_i) = \varphi(\boldsymbol{\gamma})$, where $\kappa(\alpha_i) = \frac{\alpha_i}{(\bar{G}(x))^{-\alpha_i} - 1}$.

If $\boldsymbol{\alpha}, \boldsymbol{\gamma} \in \boldsymbol{D}$, from Lemma 3.2.4, we have that $\kappa(\alpha_i)$ is decreasing and convex, and from Lemma 3.2.1 (b.ii), $\varphi(\boldsymbol{\alpha})$ is Schur-convex in $\boldsymbol{\alpha}$ whenever $\boldsymbol{\beta} \in \boldsymbol{E}_+$. Thus, if $\boldsymbol{\alpha}, \boldsymbol{\gamma} \in \boldsymbol{D}$, $\boldsymbol{\beta} \in \boldsymbol{E}_+$ and $\boldsymbol{\alpha} \overset{m}{\succeq} \boldsymbol{\gamma}$ then $X_{n:n} \geq_{rhr} Y_{n:n}$. Clearly, $\varphi(\boldsymbol{\alpha})$ is decreasing in α_i, then from Lemma 3.2.3, $\boldsymbol{\alpha}, \boldsymbol{\gamma} \in \boldsymbol{D}$, $\boldsymbol{\beta} \in \boldsymbol{E}_+$ and $\boldsymbol{\alpha} \overset{w}{\succeq} \boldsymbol{\gamma}$ implies $X_{n:n} \geq_{rhr} Y_{n:n}$.

Analogously, if $\boldsymbol{\alpha}, \boldsymbol{\gamma} \in \boldsymbol{E}$, from Lemma 3.2.4, we know that $\kappa(\alpha_i)$ is decreasing and convex, and from Lemma 3.2.2 (a.ii), $\varphi(\boldsymbol{\alpha})$ is Schur-convex in $\boldsymbol{\alpha}$ whenever $\boldsymbol{\beta} \in \boldsymbol{D}_+$. Thus, if $\boldsymbol{\alpha}, \boldsymbol{\gamma} \in \boldsymbol{E}$, $\boldsymbol{\beta} \in \boldsymbol{D}_+$ and $\boldsymbol{\alpha} \overset{m}{\succeq} \boldsymbol{\gamma}$

then $X_{n:n} \geq_{rhr} Y_{n:n}$. Clearly, $\varphi(\boldsymbol{\alpha})$ is decreasing, then from Lemma 3.2.3, if $\boldsymbol{\alpha}, \boldsymbol{\gamma} \in \boldsymbol{E}$, $\boldsymbol{\beta} \in \boldsymbol{D}_+$ and $\boldsymbol{\alpha} \overset{w}{\succeq} \boldsymbol{\gamma}$ then $X_{n:n} \geq_{rhr} Y_{n:n}$. □

The following counterexample shows that Theorem 3.3.2 does not hold under the conditions $\boldsymbol{\beta} \in \boldsymbol{E}+, \boldsymbol{\alpha}, \boldsymbol{\gamma} \in \boldsymbol{D}$, even $\boldsymbol{\alpha} \overset{m}{\succeq} \boldsymbol{\gamma}$.

Counterexample 3.3.2. *Let X_i and Y_i be two random variables, with $X_i \sim$EGNH$(\alpha_i, \beta_i, G(\cdot))$ and $Y_i \sim$EGNH$(\gamma_i, \beta_i, G(\cdot))$, for $\boldsymbol{\alpha} = (0.30, 0.13, 0.08) \in \boldsymbol{D}$, $\boldsymbol{\gamma} = (0.215, 0.215, 0.080) \in \boldsymbol{D}$ and $\boldsymbol{\beta} = (2, 2.2, 3.5) \in \boldsymbol{E}_+$, clearly $\boldsymbol{\alpha} \overset{m}{\succeq} \boldsymbol{\gamma}$. Substituting $x > 0$ by $\mathrm{e}^{-x} > 0$, implies that $F_{3:3}/H_{3:3}$ is not increasing as can be seem in Figure 3.3.*

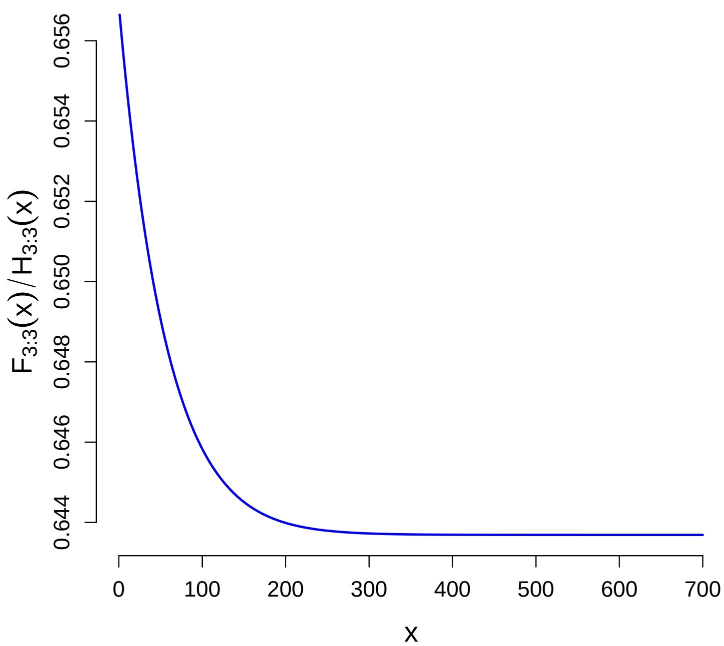


Figure 3.3: Plot of $F_{3:3}/H_{3:3}$ for $\boldsymbol{\alpha}, \boldsymbol{\gamma} \in \boldsymbol{D}$ and $\boldsymbol{\beta} \in \boldsymbol{E}_+$

Next counterexample shows that it is not possible to extend the result of Theorem 3.3.2 even if $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ are ordered in the majorization order sense.

Counterexample 3.3.3. *For $0 < x < 1$, let $X_i \sim$EGKw$(\alpha_i, \beta_i, G(\cdot))$ and $Y_i \sim$EGKw$(\gamma_i, \beta_i, G(\cdot))$ where $\boldsymbol{\alpha}, \boldsymbol{\gamma} \in \boldsymbol{D}$ and $\boldsymbol{\beta} \in \boldsymbol{E}_+$. Let $\boldsymbol{\alpha} = (5.2, 3.1, 1)$, $\boldsymbol{\beta} = (2, 3, 4)$ and $\boldsymbol{\gamma} = (4.2, 4.1, 1)$, we notice that $\boldsymbol{\alpha} \overset{m}{\succeq} \boldsymbol{\gamma}$ and $X_{3:3} \not\geq_{lr} Y_{3:3}$.*

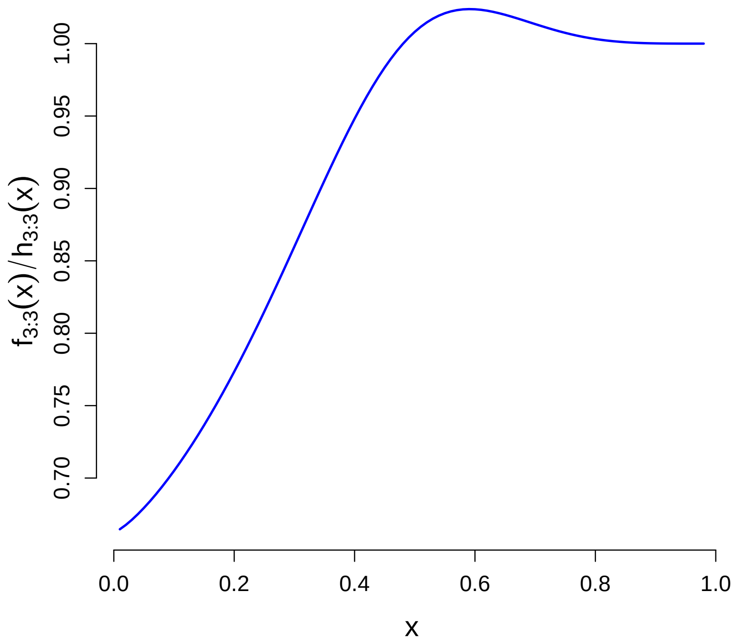


Figure 3.4: Plot of $f_{3:3}/h_{3:3}$ for $\boldsymbol{\alpha}, \boldsymbol{\gamma} \in \boldsymbol{D}$ and $\boldsymbol{\beta} \in \boldsymbol{E}_+$

The following result establishes that the restrictions on the parameters of Theorem 3.3.2 hold in the next multiple outlier case.

Theorem 3.3.3. *For $i = 1, 2, \ldots, n$, let X_i, Y_i be two sets of independent random variables following multiple outlier EG class, such that $X_i \sim EG(\alpha, \beta_i, G(\cdot))$ and $Y_i \sim EG(\gamma, \beta_i, G(\cdot))$ for $i{=}1, 2, \ldots, p$, and $X_i \sim EG(\alpha^*, \beta_i^*, G(\cdot))$ and $Y_i \sim EG(\alpha^*, \beta_i^*, G(\cdot))$ for $i{=}p+1, p+2, \ldots, p+q(=n)$. Then $(\underbrace{\alpha, \alpha, \ldots, \alpha}_{p}, \underbrace{\alpha^*, \alpha^*, \ldots, \alpha^*}_{q}) \overset{w}{\succeq} (\underbrace{\gamma, \gamma, \ldots, \gamma}_{p}, \underbrace{\alpha^*, \alpha^*, \ldots, \alpha^*}_{q})$, implies $X_{n:n} \geq_{rhr} Y_{n:n}$.*

Proof. From (3.4) and (3.5), we know that

$$\tilde{r}_{n:n}(x) = \tilde{r}_G(x) \left(\sum_{i=1}^{p} \beta_i \kappa(\alpha) + \sum_{i=p+1}^{n} \beta_i^* \kappa(\alpha^*) \right) \quad \text{and} \quad \tilde{s}_{n:n}(x) = \tilde{r}_G(x) \left(\sum_{i=1}^{p} \beta_i \kappa(\gamma) + \sum_{i=p+1}^{n} \beta_i^* \kappa(\alpha^*) \right).$$

Moreover, $(\underbrace{\alpha, \alpha, \ldots, \alpha}_{p}, \underbrace{\alpha^*, \alpha^*, \ldots, \alpha^*}_{q}) \overset{w}{\succeq} (\underbrace{\gamma, \gamma, \ldots, \gamma}_{p}, \underbrace{\alpha^*, \alpha^*, \ldots, \alpha^*}_{q})$ if, and only if $\alpha \leq \gamma$ and ($\gamma \leq \alpha^*$ or $\alpha^* \leq \alpha$). Observe that in these cases $\tilde{r}_{n:n}(x) \geq \tilde{s}_{n:n}(x)$, consequently $X_{n:n} \geq_{rhr} Y_{n:n}$.

□

Now, we study the likelihood ratio order between maximum statistic orders, following multiple outlier model.

Theorem 3.3.4. *Let X_i and Y_i with i=$1, 2, \ldots, n$, be two sets of mutually independent random variables following multiple outlier EG class, such that X_i∼EG$(\alpha, \beta_i, G(\cdot))$ and Y_i∼EG$(\gamma, \beta_i, G(\cdot))$ for i=$1, 2, \ldots, p$, X_i∼EG$(\alpha^*, \beta_i^*, G(\cdot))$ and Y_i∼EG$(\gamma^*, \beta_i^*, G(\cdot))$ for i=$p+1, p+2, \ldots, p+q(=n)$. Suppose that $\alpha, \alpha^* \in \boldsymbol{D}$, $\gamma, \gamma^* \in \boldsymbol{D}$ and $\beta, \beta^* \in \boldsymbol{E}_+$ holds.*

Then $(\underbrace{\alpha, \alpha, \ldots, \alpha}_{p}, \underbrace{\alpha^*, \alpha^*, \ldots, \alpha^*}_{q}) \overset{m}{\succeq} (\underbrace{\gamma, \gamma, \ldots, \gamma}_{p}, \underbrace{\gamma^*, \gamma^*, \ldots, \gamma^*}_{q})$ *implies* $X_{n:n} \geq_{lr} Y_{n:n}$.

Proof. We need to prove

$$\frac{f_{n:n}(x)}{h_{n:n}(x)} = \frac{\tilde{r}_{n:n}(x)}{\tilde{s}_{n:n}(x)} \frac{F_{n:n}(x)}{H_{n:n}(x)} \quad \text{is increasing in } x.$$

Here $\frac{F_{n:n}(x)}{H_{n:n}(x)}$ is increasing by Theorem (3.3.2). It is sufficient to show that $\psi(x) = \frac{\tilde{r}_{n:n}(x)}{\tilde{s}_{n:n}(x)}$ is increasing in x.

From (4), we obtain

$$\begin{aligned} \tilde{r}_{n:n}(x) &= r_G(x) \left(\frac{p\alpha}{[1-G(x)]^{-\alpha} - 1} \sum_{i=1}^{p} \beta_i + \frac{q\alpha^*}{[1-G(x)]^{-\alpha^*} - 1} \sum_{i=p+1}^{n} \beta_i \right) \\ &= r_G(x) \left(p\kappa(\alpha) \sum_{i=1}^{p} \beta_i + q\kappa(\alpha^*) \sum_{i=p+1}^{n} \beta_i \right), \end{aligned}$$

where $\kappa(\alpha) = \dfrac{\alpha}{(1-G(x))^{-\alpha} - 1}$. Thus,

$$\psi(x) = \frac{p\kappa(\alpha) \sum_{i=1}^{p} \beta_i + q\kappa(\alpha^*) \sum_{i=p+1}^{n} \beta_i}{p\kappa(\gamma) \sum_{i=1}^{p} \beta_i + q\kappa(\gamma^*) \sum_{i=p+1}^{n} \beta_i}.$$

Differentiating with respect to x,

$$\begin{aligned} \psi'(x) \overset{sign}{=} & \left(p\kappa'(\alpha) \sum_{i=1}^{p} \beta_i + q\kappa'(\alpha^*) \sum_{i=p+1}^{n} \beta_i \right) \left(p\kappa(\gamma) \sum_{i=1}^{p} \beta_i + q\kappa(\gamma^*) \sum_{i=p+1}^{n} \beta_i \right) \\ - & \left(p\kappa(\alpha) \sum_{i=1}^{p} \beta_i + q\kappa(\alpha^*) \sum_{i=p+1}^{n} \beta_i \right) \left(p\kappa'(\gamma) \sum_{i=1}^{p} \beta_i + q\kappa'(\gamma^*) \sum_{i=p+1}^{n} \beta_i \right), \end{aligned}$$

where κ' is the partial derivate of κ with respect to x.

Then $\psi'(x)$ is increasing if

$$\frac{p\kappa'(\alpha)\sum_{i=1}^{p}\beta_i + q\kappa'(\alpha^*)\sum_{i=p+1}^{n}\beta_i}{p\kappa(\alpha)\sum_{i=1}^{p}\beta_i + q\kappa(\alpha^*)\sum_{i=p+1}^{n}\beta_i} \geq \frac{p\kappa'(\gamma)\sum_{i=1}^{p}\beta_i + q\kappa'(\gamma^*)\sum_{i=p+1}^{n}\beta_i}{p\kappa(\gamma)\sum_{i=1}^{p}\beta_i + q\kappa(\gamma^*)\sum_{i=p+1}^{n}\beta_i},$$

that is, if the function

$$\Psi(\boldsymbol{\alpha}, \boldsymbol{\alpha}^*) = \frac{p\kappa'(\alpha)\sum_{i=1}^{p}\beta_i + q\kappa'(\alpha^*)\sum_{i=p+1}^{n}\beta_i}{p\kappa(\alpha)\sum_{i=1}^{p}\beta_i + q\kappa(\alpha^*)\sum_{i=p+1}^{n}\beta_i},$$

is Schur-Convex in $(\boldsymbol{\alpha}, \boldsymbol{\alpha}^*)$.

Let us denote $\beta = \sum_{i=1}^{n}\beta_i$, $\beta^* = \sum_{i=1}^{n}\beta_i^*$, and let $\kappa'(\alpha) = \kappa(\alpha)\phi(\alpha)r_G(x)$ where $\phi(\alpha) = -\dfrac{\alpha}{1-(\bar{G}(x))^{\alpha}}$.

Thus, we have

$$\Psi(\boldsymbol{\alpha}, \boldsymbol{\alpha}^*) = r_G(x)\frac{p\kappa(\alpha)\phi(\alpha)\beta + q\kappa(\alpha^*)\phi(\alpha^*)\beta^*}{p\kappa(\alpha)\beta + q\kappa(\alpha^*)\beta^*}.$$

On differentiating $\Psi(\boldsymbol{\alpha}, \boldsymbol{\alpha}^*)$ with respect to α, and let $\phi_1(\alpha) = \kappa(\alpha)\phi'(\alpha)$, we obtain

$$\begin{aligned}
\frac{\partial\Psi}{\partial\alpha} &= r_G(x)\frac{[p\kappa'(\alpha)\phi(\alpha)\beta + p\kappa(\alpha)\phi'(\alpha)\beta]\,[p\kappa(\alpha)\beta + q\kappa(\alpha^*)\beta^*] - [p\kappa'(\alpha)\beta]\,[p\kappa(\alpha)\phi(\alpha)\beta + q\kappa(\alpha^*)\phi(\alpha^*)\beta^*]}{(p\kappa(\alpha)\beta + q\kappa(\alpha^*)\beta^*)^2} \\
&= p\left(r_G(x)\frac{[q\kappa'(\alpha)\beta\kappa(\alpha^*)\beta^*\{\phi(\alpha) - \phi(\alpha^*)\} + \kappa(\alpha)\phi'(\alpha)\beta\{p\kappa(\alpha)\beta + q\kappa(\alpha^*)\beta^*\}]}{(p\kappa(\alpha)\beta + q\kappa(\alpha^*)\beta^*)^2}\right) \\
&= p\left(r_G(x)\frac{[q\kappa'(\alpha)\beta\kappa(\alpha^*)\beta^*\{\phi(\alpha) - \phi(\alpha^*)\} + \phi_1(\alpha)\beta\{p\kappa(\alpha)\beta + q\kappa(\alpha^*)\beta^*\}]}{(p\kappa(\alpha)\beta + q\kappa(\alpha^*)\beta^*)^2}\right).
\end{aligned}$$

With a similar expression for α^*, and let $\phi_1(\alpha^*) = \kappa(\alpha^*)\phi'(\alpha^*)$, we have

$$\begin{aligned}
\frac{\partial\Psi}{\partial\alpha^*} &= r_G(x)\frac{[q\kappa'(\alpha^*)\phi(\alpha^*)\beta^* + q\kappa(\alpha^*)\phi'(\alpha^*)\beta^*]\,[p\kappa(\alpha)\beta + q\kappa(\alpha^*)\beta^*] - [q\kappa'(\alpha^*)\beta^*]\,[p\kappa(\alpha)\phi(\alpha)\beta + q\kappa(\alpha^*)\phi(\alpha^*)\beta^*]}{(p\kappa(\alpha)\beta + q\kappa(\alpha^*)\beta^*)^2} \\
&= q\left(r_G(x)\frac{[p\kappa'(\alpha^*)\beta^*\kappa(\alpha)\beta\{\phi(\alpha^*) - \phi(\alpha)\} + \kappa(\alpha^*)\phi'(\alpha^*)\beta^*\{q\kappa(\alpha^*)\beta^* + p\kappa(\alpha)\beta\}]}{(p\kappa(\alpha)\beta + q\kappa(\alpha^*)\beta^*)^2}\right) \\
&= q\left(r_G(x)\frac{[p\kappa'(\alpha^*)\beta^*\kappa(\alpha)\beta\{\phi(\alpha^*) - \phi(\alpha)\} + \phi_1(\alpha^*)\beta^*\{q\kappa(\alpha^*)\beta^* + p\kappa(\alpha)\beta\}]}{(p\kappa(\alpha)\beta + q\kappa(\alpha^*)\beta^*)^2}\right).
\end{aligned}$$

Hence arise:

$$\begin{aligned}
\frac{\partial\Psi}{\partial\alpha}-\frac{\partial\Psi}{\partial\alpha^*} &= p\left(r_G(x)\frac{[q\kappa'(\alpha)\beta\kappa(\alpha^*)\beta^*\{\phi(\alpha)-\phi(\alpha^*)\}+\phi_1(\alpha)\beta\{p\kappa(\alpha)\beta+q\kappa(\alpha^*)\beta^*\}]}{(p\kappa(\alpha)\beta+q\kappa(\alpha^*)\beta^*)^2}\right)\\
&\quad -q\left(r_G(x)\frac{[p\kappa'(\alpha^*)\beta^*\kappa(\alpha)\beta\{\phi(\alpha^*)-\phi(\alpha)\}+\phi_1(\alpha^*)\beta^*\{q\kappa(\alpha^*)\beta^*+p\kappa(\alpha)\beta\}]}{(p\kappa(\alpha)\beta+q\kappa(\alpha^*)\beta^*)^2}\right)\\
&\overset{\text{sign}}{=} \left[\{\phi(\alpha)-\phi(\alpha^*)\}\left\{pq\kappa'(\alpha)\beta\kappa(\alpha^*)\beta^*+qp\kappa'(\alpha^*)\beta^*\kappa(\alpha)\beta\right\}\right]+\\
&\quad [\{p\kappa(\alpha)\beta+q\kappa(\alpha^*)\beta^*\}\{p\phi_1(\alpha)\beta-q\phi_1(\alpha^*)\beta^*\}]. \qquad (3.6)
\end{aligned}$$

Now, ϕ is decreasing with respect to α, and $\kappa'\leq 0$, observe that the first bracketed term of (3.6) is not negative. Note that, the second bracketed term of (3.6) also is positive, thus ϕ_1, is increasing, it follows that $\frac{\partial\Psi}{\partial\alpha}-\frac{\partial\Psi}{\partial\alpha^*}\geq 0$. Thus, Ψ is Schur convex in $\boldsymbol{\alpha}$. Hence, $\boldsymbol{\alpha}\overset{m}{\succeq}\boldsymbol{\gamma}$ implies $\psi\geq 0$, proving the result.

□

We have the following example which verify the conditions of the result given in the previous theorem.

Example 3.3.2. *Let X_i, Y_i following the EGF. Also, let $\alpha=1.2$, $\alpha^*=2.3$, with $(p,q)=(4,3)$, $\beta=3.9$, $\beta^*=2.1$, $\gamma=1.5$, $\gamma^*=1.74$, and $(p,q)=(2,5)$. Based on the Figure 3.5 observe that $X_{n:n}\geq_{lr}Y_{n:n}$,*

$$(\underbrace{\alpha,\alpha,\ldots,\alpha}_{p},\underbrace{\alpha^*,\alpha^*,\ldots,\alpha^*}_{q})\overset{m}{\succeq}(\underbrace{\gamma,\gamma,\ldots,\gamma}_{p},\underbrace{\gamma^*,\gamma^*,\ldots,\gamma^*}_{q}).$$

The following theorems show what happens when we have two different baselines in class EG and we established conditions on the parameters. The results are given in the usual stochastic order and our study reveals the following.

Let X and Y two random variables with $G_1(\cdot)$ and $G_2(\cdot)$ continous cdf, respectively. Let $X_i\sim EG(\alpha_i,\beta_i,G_1(\cdot))$ and $Y_i\sim EG(\gamma_i,\delta_i,G_2(\cdot))$ two sets of mutually independent random variables and i=$1,2,\ldots,n$.

Theorem 3.3.5. *Let X_i and Y_i be two sets of mutually independent random variables. $X_i\sim EG(\alpha_i,\beta_i,G_1(\cdot))$ and $Y_i\sim EG(\gamma_i,\beta_i,G_2(\cdot))$ with i=$1,2,\ldots,n$. Suppose that $\boldsymbol{\alpha},\boldsymbol{\gamma}\in\boldsymbol{E}$ and $\boldsymbol{\beta}\in\boldsymbol{D}_+$ or $\boldsymbol{\alpha},\boldsymbol{\gamma}\in\boldsymbol{D}$ and $\boldsymbol{\beta}\in\boldsymbol{E}_+$ with $\boldsymbol{\alpha}\overset{m}{\succeq}\boldsymbol{\gamma}$, then $X\geq_{st}Y$ implies $X_{n:n}\geq_{st}Y_{n:n}$.*

Proof. Considering $Z_i\sim$EG$(\gamma_i,\beta_i,G_1(\cdot))$ another random variable. By the Theorem 3.3.2 we can see that $\boldsymbol{\alpha},\boldsymbol{\gamma}\in\boldsymbol{E}$ and $\boldsymbol{\beta}\in\boldsymbol{D}_+$ with $\boldsymbol{\alpha}\overset{w}{\succeq}\boldsymbol{\gamma}$, then $X_{n:n}\geq_{rhr}Z_{n:n}$ which implies $X_{n:n}\geq_{st}Z_{n:n}$. Let $\bar{G}(\cdot)=1-G(\cdot)$. Then by using Definition 3.2.2,

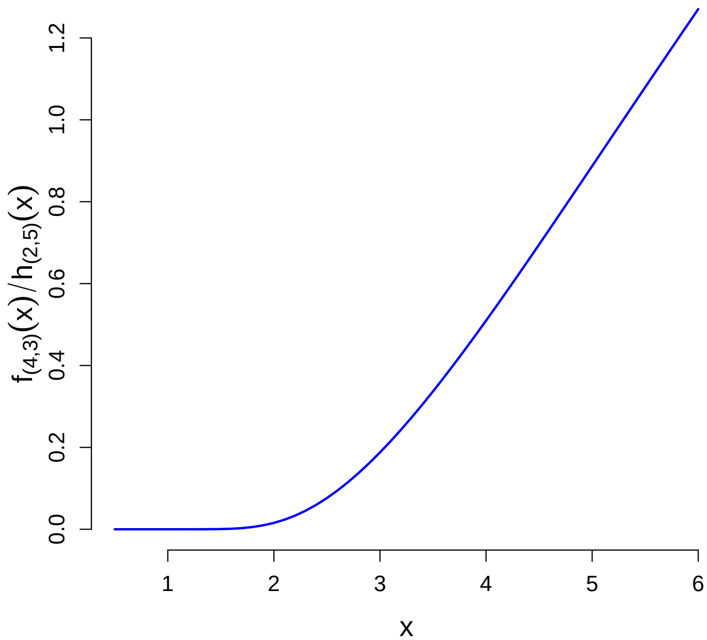


Figure 3.5: Plot of Likelihood ratio for $f_{(4,3)}/h_{(2,5)}$

$$\begin{aligned} \prod_{i=1}^{n} 1-[1-\{1-G_1(\cdot)\}^{\alpha_i}]^{\beta_i} &\geq \prod_{i=1}^{n} 1-[1-\{1-G_1(\cdot)\}^{\gamma_i}]^{\beta_i}\,, \\ \prod_{i=1}^{n} [1-\{1-G_1(\cdot)\}^{\alpha_i}]^{\beta_i} &\leq \prod_{i=1}^{n} [1-\{1-G_1(\cdot)\}^{\gamma_i}]^{\beta_i}\,. \end{aligned} \tag{3.7}$$

Now, $X \geq_{st} Y$ implies $\bar{G}_1(\cdot) \geq \bar{G}_2(\cdot)$, that is

$$\begin{aligned} \left[1-\left\{\bar{G}_1(\cdot)\right\}^{\gamma_i}\right]^{\beta_i} &\leq \left[1-\left\{\bar{G}_2(\cdot)\right\}^{\gamma_i}\right]^{\beta_i}\,, \\ \prod_{i=1}^{n} 1-\left[1-\left\{\bar{G}_1(\cdot)\right\}^{\gamma_i}\right]^{\beta_i} &\geq \prod_{i=1}^{n} 1-\left[1-\left\{\bar{G}_2(\cdot)\right\}^{\gamma_i}\right]^{\beta_i}\,. \end{aligned} \tag{3.8}$$

So, from equation (3.7) and (3.8)

$$\prod_{i=1}^{n} 1-\left[1-\left\{\bar{G}_1(\cdot)\right\}^{\alpha_i}\right]^{\beta_i} \geq \prod_{i=1}^{n} 1-\left[1-\left\{\bar{G}_1(\cdot)\right\}^{\gamma_i}\right]^{\beta_i} \geq \prod_{i=1}^{n} 1-\left[1-\left\{\bar{G}_2(\cdot)\right\}^{\gamma_i}\right]^{\beta_i}\,,$$

implying $X_{n:n} \geq_{st} Y_{n:n}$.

□

Theorem 3.3.6. *Let X_i and Y_i be two sets of mutually independent random variables with $X_i \sim EG(\alpha_i, \beta_i, G_1(\cdot))$ and $Y_i \sim EG(\alpha_i, \delta_i, G_2(\cdot))$ with $i = 1, 2, \ldots, n$. If $\boldsymbol{\beta} \overset{m}{\succeq} \boldsymbol{\delta}$ and*

(i) $\boldsymbol{\alpha} \in \boldsymbol{E}_+$, $\boldsymbol{\beta}, \boldsymbol{\delta} \in \boldsymbol{D}$ or $\boldsymbol{\alpha} \in \boldsymbol{D}_+$, $\boldsymbol{\beta}, \boldsymbol{\delta} \in \boldsymbol{E}$ then $X \geq_{st} Y$ implies $X_{n:n} \geq_{st} Y_{n:n}$;

(ii) $\boldsymbol{\alpha} \in \boldsymbol{D}_+$, $\boldsymbol{\beta}, \boldsymbol{\delta} \in \boldsymbol{D}$ or $\boldsymbol{\alpha} \in \boldsymbol{E}_+$, $\boldsymbol{\beta}, \boldsymbol{\delta} \in \boldsymbol{E}$ then $X \leq_{st} Y$ implies $X_{n:n} \leq_{st} Y_{n:n}$.

Proof. Let us consider the random variable Z_i as defined in Theorem 3.3.5, by Theorem 3.3.1 and the same process from Theorem 3.3.5 the desired results follow immediately. □

3.4 A Possible Application

In this section similar as Kayal (2019), without loss of generality a possible application of the theoretical results is provided. Suppose there are parallel systems consisting of n components in a motor Boeing, commonly used in reliability theory. The components are assumed to function independently. The system components works if at least one of the components works. The failure times of the components could assumed to follow the EG class. We are interested to compare the performance of the parallel systems stochastically considering the EG class.

As an application, we could see the operation of two Boeing engines, because they work as parallel systems and verify the power and the life time in their operation, see Figure 3.6 and Figure 3.7 (source:https://br.pinterest.com/AyameDesign/industrial). Theorem 3.3.2 guarantees that, for parallel systems of components following the EG class distribution with a common β shape parameter, the weakly majorized parameter α leads to a longer system life in the sense of the order of the reversed hazard rate when the shape parameter is increasing.

Here, we consider the complex operation of a motor Boeing. Obviously, the operation of that depends on multiple components, but it serves to illustrate how the results could be applied in this field. For the purposes of this thesis, the numerical application escapes from our hands because many tools are needed for a more applied part.

Figure 3.6: Motor Boeing

Figure 3.7: Part of Motor Boeing

3.5 Conclusion and Future Works

In this research, we studied stochastic comparisons of parallel system formed by components having independent lifetimes, based on the Exponentiated Generalized class. In the results, we use the theory of majorization and stochastic order. We derive some conditions under the parameters to give some results to respect reversed hazard rate and stochastic order. Also, for the case of multiple outlier we have results comparing by the same baseline. In addition of the examples to illustrate the research, we had the caution of showed some failure through counterexamples. For future works, we recommend to study the possible fields of application and apply the results to real databases in reliability or other areas. Also, to study the majorization theory in other distribution functions.

Chapter 4

Inferential Results For Transmuted Inverse Weibull Distribution

Abstract

Different measures of Goodness-of-Fit yield information to describe the fits of models to the data. This research constructed qualitative and quantitative fit measures for Transmuted Inverse Weibull distribution. To develop these Goodness-of-Fit measures, we study some properties of that distribution: we present the Mellin Transform, Log-Moments, and Log-Cumulants. Then, we discuss estimation methods for the model's parameters, such as Moments, Maximum Likelihood, and the one based on the Log-Cumulants method. The last method mentioned is proposed to estimate the parameters of the distribution. The model is applied to three survival datasets to verify the quality of our estimation methods and Goodness-of-Fit measures.

Keywords: Parameter Estimation, Log-Cumulants, Goodness-of-Fit, Confidence Ellipse.

4.1 Introduction

Lifetime probability distributions has been used in different context such as reliability (Kececioglu, 2002; Meeker et al., 2021), life data (Lee et al., 2007), modelling failure and engineering (Pescim et al., 2013), among others. Researchers generally introduce parameters or/and generators to obtain more new/flexible models. In this chapter, we use a distribution that provides a particular transmuting function for constructing tractable models.

The Weibull distribution was introduced by Weibull et al. (1951) and also mentioned in Rinne (2008). This distribution is mainly used in quality control Nelson (1979) and studied in Hallinan Jr (1993), reliability and applied statistic (Nelson, 1985; Ambrožič and Gorjan, 2011), and hidrology (Singh, 1987; Saboor et al., 2016), among others. Nevertheless, in some situations which it is not possible to fit simple models to the data, it is necessary to create new models that supply this fact, by adding parameters or base distribution, to guarantee the flexibility of the model and better fits.

Keller et al. (1982) derived four alternative failure models based on physical considerations, particularly the two-parameter model known as the Inverse Weibull model. Such distribution plays an essential role in modeling failure rates which are extremely important in biological, reliability, and survival analysis studies. Subsequently, some works generalize or modify the inverse Weibull, for example, De Gusmao et al. (2011), Khan and King (2012), Khan and King (2016) and Jan et al. (2017).

An excellent idea of a generalization model where the distribution is derived using the quadratic rank transmutation map is intended to motivate our investigation; it was giving by Shaw and Buckley (2009). In this context, Khan and King (2014) introduce the three-parameter Transmuted Inverse Weibull (TIW) distribution, evidencing the Transmuted Inverse Exponential, Transmuted Inverse Rayleigh and Inverse Weibull distributions as sub-models.

Merovci et al. (2013) created a New Generalized Inverse Weibull distribution. Al-Omari (2018) developed new acceptance sampling plan based on the Transmuted Generalized Inverse Weibull distribution. AL-Kadim and Mohammed (2017) proposed a cubic Transmuted Weibull distribution and discuss some submodels or special cases chosen particular parameters values and giving theoretical results.

Recently, Rahman et al. (2020) studied a detailed review of the transmuted families of distributions and Dey et al. (2021) reviewed a complete list of transmuted distributions.

Now, let us mention the Mellin Transform (MT); it surges in a mathematical context, the Finnish mathematician R.H.Mellin (1854-1933) was the first to formulate that, and it has been applied in different fields of engineering. Butzer and Jansche (1997) show an application to the partial differential equations as a particular use of the differential properties. Nicolas (2002) presents an interesting application based on graphic representation using Log-Cumulants (LC) of order two and three which leads to graph the LC diagram. Recently, Jain et al. (2021) introduce a study that they called a (p, q)-Mellin transform and its corresponding convolution and inversion. They solve some integral equations in terms of applications of the (p, q)-Mellin transform.

So far, no developed measures of Goodness-of-Fit (GoF) have been applied to the TIW distribution, making it difficult to select this model in some cases. Finally, there are different techniques in the literature to fit the models by using criteria and measures for different distributions, known as GoF. In this line, Pearson (1895) provides an excellent tool for choosing a model known as the Pearson diagram, which uses skewness and kurtosis. However, Nicolas (2002) considered this method sometimes analytically intractable. Hence, build the LC diagram for some classical models illustrate SAR image data. Vasconcelos et al. (2021) recently constructed LC diagrams and derived confidence ellipses for the LC for the beta-G class based on Hotelling's T^2 statistic.

In this thesis, we study the MT to aim to establish new GoF measures for the TIW distribution and compare the Maximum Likelihood, Moments, and LC estimates. Using the T^2 statistic and the confidence ellipses for hypothesis testing, we illustrate the performance of the measures treated in some datasets.

This work, aims to give some inferential results by using new GoF method. To accomplish this, we provide a study specially of the MT, LC and Log-Moments. The Chapter is organized as follows: Section 4.2 presents a brief overview of TIW distribution and also the theory of parameters estimation. Section 4.3 proposes a second kind statistic for the TIW distribution. Section 4.4 offers Log-Cumulants and T^2 statistic. Section 4.5 a brief simulation. Section 4.6 shows an application to real dataset and finally, Section 4.7 presents the conclusions.

4.2 Transmuted Inverse Weibull distribution

In this section, let us introduce some aspects of how the TIW could be created.

A random variable Y is said to follow a Weibull distribution (Weibull et al., 1951; Rinne, 2008) with parameters $\beta > 0$ and $\eta > 0$, the scale and shape parameters, respectively, if its CDF and PDF are,

$$G_Y(y) = 1 - \mathrm{e}^{-\left(\frac{y}{\beta}\right)^{\eta}} \qquad \text{and} \qquad g_Y(y) = \frac{\eta}{\beta}\left(\frac{y}{\beta}\right)^{\eta-1} \mathrm{e}^{-\left(\frac{y}{\beta}\right)^{\eta}}, \quad y \geq 0.$$

Let Z be a random variable following Inverse Weibull distribution, say $Z \sim IW(z; \beta, \eta)$, with parameters $\beta > 0$ and $\eta > 0$, the scale and shape parameters, respectively. The CDF and PDF, respectively, are given by (Keller et al., 1982), see also Khan et al. (2008):

$$G_Z(z) = \mathrm{e}^{-\frac{1}{\eta}\left(\frac{1}{z}\right)^{\beta}}, \qquad \text{and} \qquad g_Z(z) = \left(\frac{\beta}{\eta}\right)\left(\frac{1}{z}\right)^{\beta+1} \mathrm{e}^{-\frac{1}{\eta}\left(\frac{1}{z}\right)^{\beta}}, \quad z > 0.$$

Transmutation is the functional composition of the CDF of one distribution with the inverse cumulative distribution function (quantile function) of another (Rahman et al., 2020). A random variable W is said to follow transmuted distribution, see (Shaw and Buckley, 2009), if its CDF is given by

$$F_W(w) = (1 + \lambda)G(w) - \lambda G(w)^2, \quad |\lambda| \leq 1, \tag{4.1}$$

where λ offers more flexibility in the distribution and $G(w)$ is the CDF of the baseline distribution.

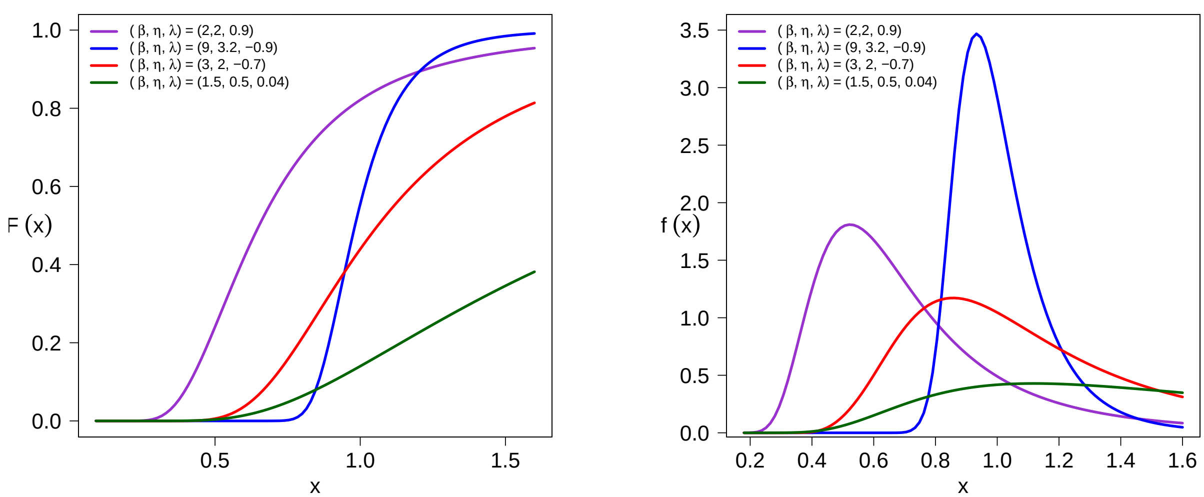
Finally, this distribution arises like a new reliability model. From (4.1), a random variable X is said to follow the TIW distribution denoted by, $X \sim TIW(x; \beta, \eta, \lambda)$ if the CDF and PDF of X are given by

$$F_X(x) = (1+\lambda)\,\mathrm{e}^{-\frac{1}{\eta}\left(\frac{1}{x}\right)^{\beta}} - \lambda \left(\mathrm{e}^{-\frac{1}{\eta}\left(\frac{1}{x}\right)^{\beta}}\right)^2,$$

$$f_X(x) = \left(\frac{\beta}{\eta}\right)\left(\frac{1}{x}\right)^{\beta+1} \mathrm{e}^{-\frac{1}{\eta}\left(\frac{1}{x}\right)^{\beta}} \left(1+\lambda-2\lambda\,\mathrm{e}^{-\frac{1}{\eta}\left(\frac{1}{x}\right)^{\beta}}\right), \qquad x>0.$$

Where $\beta, \eta > 0$, and $|\lambda| \leq 1$, the shape, scale and transmuting parameters, respectively (Khan and King, 2014).

Figure 4.1 illustrates the flexibility of the CDF and PDF for selected parameter values. Note that, when $\lambda = 0.04$ the density is a monotonic function and when $\beta > 1.5$ becomes unimodal. When x tends to infinity the density tends to zero. Now, when λ is close to 1 has a leptokurtic form already in the case of $\lambda < 1$ as it decreases has a platykurtic shape.



(a) (b)

Figure 4.1: Transmuted Inverse Weibull CDF (a) and PDF (b) for different values of β, η, λ

Using u a random number from zero to one. The quantile function from TIW, by solving $F_X(x) \leq u$ is

$$x = \left(-\eta \log\left(\frac{(1+\lambda) - \sqrt{(1+\lambda)^2 - 4\lambda u}}{2\lambda}\right)\right)^{\frac{-1}{\beta}},$$

step by step, the equation above is

$$
\begin{aligned}
F_X(x) &= u \\
(1+\lambda)\,\mathrm{e}^{-\frac{x^{-\beta}}{\eta}} - \lambda\left(\mathrm{e}^{-\frac{x^{-\beta}}{\eta}}\right)^2 &= u \\
\frac{4(1+\lambda)\lambda\,\mathrm{e}^{-\frac{x^{-\beta}}{\eta}} - 4\lambda^2\left(\mathrm{e}^{-\frac{x^{-\beta}}{\eta}}\right)^2}{4\lambda} &= u \\
(1+\lambda)^2 - (1+\lambda)^2 + 4(1+\lambda)\left(\lambda\,\mathrm{e}^{-\frac{x^{-\beta}}{\eta}}\right) - 4\lambda^2\left(\mathrm{e}^{-\frac{x^{-\beta}}{\eta}}\right)^2 &= 4\lambda u \\
(1+\lambda)^2 - 4(1+\lambda)\left(\lambda\,\mathrm{e}^{-\frac{x^{-\beta}}{\eta}}\right) + 4\lambda^2\left(\mathrm{e}^{-\frac{x^{-\beta}}{\eta}}\right)^2 &= (1+\lambda)^2 - 4\lambda u \\
\left((1+\lambda) - 2\lambda\,\mathrm{e}^{-\frac{x^{-\beta}}{\eta}}\right)^2 &= (1+\lambda)^2 - 4\lambda u \\
(1+\lambda) - 2\lambda\,\mathrm{e}^{-\frac{x^{-\beta}}{\eta}} &= \sqrt{(1+\lambda)^2 - 4\lambda u} \\
\frac{(1+\lambda) - \sqrt{(1+\lambda)^2 - 4\lambda u}}{2\lambda} &= \mathrm{e}^{-\frac{x^{-\beta}}{\eta}} \\
-\eta\log\left(\frac{(1+\lambda) - \sqrt{(1+\lambda)^2 - 4\lambda u}}{2\lambda}\right) &= x^{-\beta} \\
\left(-\eta\log\left(\frac{(1+\lambda) - \sqrt{(1+\lambda)^2 - 4\lambda u}}{2\lambda}\right)\right)^{-\frac{1}{\beta}} &= x.
\end{aligned}
$$

Using $u = 0.5$, we obtain the median of the TIW distribution. In practice, this is the life at which at least 50% of the units will be expected to fail.

The r-th moment from $X \sim TIW(x; \beta, \eta, \lambda)$ is given as follows

$$\mathbb{E}\left(X^r\right) = \eta^{\,-\frac{r}{\beta}}\Gamma\left(1 - \frac{r}{\beta}\right)\left(1 + \lambda - \lambda 2^{\frac{r}{\beta}}\right), \tag{4.2}$$

where $\Gamma(\cdot)$ is the Gamma function.

To estimate the parameters of TIW distribution, we present in this section two methods: the Maximum Likelihood and Moments.

4.2.1 Method of Maximum Likelihood (ML)

Let $X_1, X_2, \ldots, X_n$ be a random sample from $X \sim TIW(x;\beta,\eta,\lambda)$. The likelihood function is

$$L(\beta,\eta,\lambda;x_i) = \prod_{i=1}^{n} \frac{\beta}{\eta}\left(\frac{1}{x_i}\right)^{\beta+1} \mathrm{e}^{-\frac{1}{\eta}\left(\frac{1}{x_i}\right)^{\beta}} \left(1+\lambda-2\lambda\,\mathrm{e}^{-\frac{1}{\eta}\left(\frac{1}{x_i}\right)^{\beta}}\right), \tag{4.3}$$

where $x_i, i = 1, 2, \ldots, n$ are the observed values of the random sample.

To estimate the parameters, we find the set of values of β, η and λ that attains their maximum in (4.3).

The associated log-likelihood function of (4.3) is

$$\ell(\beta,\eta,\lambda;x_i) = n\log\frac{\beta}{\eta} + (\beta+1)\sum_{i=1}^{n}\log\left(\frac{1}{x_i}\right) - \sum_{i=1}^{n}\frac{1}{\eta}\left(\frac{1}{x_i}\right)^{\beta} + \sum_{i=1}^{n}\log\left(1+\lambda-2\lambda\,\mathrm{e}^{-\frac{1}{\eta}\left(\frac{1}{x_i}\right)^{\beta}}\right). \tag{4.4}$$

The values of $\hat{\beta}, \hat{\eta}$ and $\hat{\lambda}$ that maximize (4.4) will be the ML estimates for the parameters β, η and λ. The ML estimates are obtained by numerical methods using the `optim` function, `BFGS` method and `BB` package (Varadhan and Gilbert, 2010) in the software R Core Team (Team et al., 2013). The corresponding components of the score vector $\mathbb{S}$, by taking the partial derivates of (4.4) can be written as

$$\mathbb{S} = (\mathbb{S}_\beta, \mathbb{S}_\eta, \mathbb{S}_\lambda) = \left(\frac{\partial}{\partial\beta}\ell(\beta,\eta,\lambda;x_i), \frac{\partial}{\partial\eta}\ell(\beta,\eta,\lambda;x_i), \frac{\partial}{\partial\lambda}\ell(\beta,\eta,\lambda;x_i)\right),$$

where

$$\begin{aligned}
\frac{\partial}{\partial\beta}\ell(\beta,\eta,\lambda;x_i) &= \frac{n}{\beta} + 2\sum_{i=1}^{n}\frac{\lambda\left(\frac{1}{x_i}\right)^{\beta}\log\left(\frac{1}{x_i}\right)\mathrm{e}^{-\left(\frac{1}{\eta}\right)\left(\frac{1}{x_i}\right)^{\beta}}}{\eta\left(\lambda-2\lambda\,\mathrm{e}^{-\frac{1}{\eta}\left(\frac{1}{x_i}\right)^{\beta}}+1\right)} - \frac{1}{\eta}\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)^{\beta}\log\left(\frac{1}{x_i}\right) + \sum_{i=1}^{n}\log\left(\frac{1}{x_i}\right),\\
\frac{\partial}{\partial\eta}\ell(\beta,\eta,\lambda;x_i) &= -\frac{n}{\eta} - 2\sum_{i=1}^{n}\frac{\lambda\left(\frac{1}{x_i}\right)^{\beta}\mathrm{e}^{-\left(\frac{1}{\eta}\right)\left(\frac{1}{x_i}\right)^{\beta}}}{\eta^2\left(\lambda-2\lambda\,\mathrm{e}^{-\frac{1}{\eta}\left(\frac{1}{x_i}\right)^{\beta}}+1\right)} + \frac{1}{\eta^2}\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)^{\beta},\\
\frac{\partial}{\partial\lambda}\ell(\beta,\eta,\lambda;x_i) &= \sum_{i=1}^{n}\frac{1-2\,\mathrm{e}^{-\left(\frac{1}{\eta}\right)\left(\frac{1}{x_i}\right)^{\beta}}}{\lambda-2\lambda\,\mathrm{e}^{-\left(\frac{1}{\eta}\right)\left(\frac{1}{x_i}\right)^{\beta}}+1}.
\end{aligned}$$

Remark 4.2.1. *We consider that there is an error when writing the partial derivative of λ in Khan and King (2014) page 281, equation (36).*

4.2.2 Method of Moments (MM)

The method of moments, is an estimation method of population parameters. It involves equaling sample moments with theoretical moments. Let $X_1, X_2, \ldots, X_n$ be a random sample from $X \sim TIW(x; \beta, \eta, \lambda)$. For $r > 0$ integer, the r-th sample moment is the random variable

$$\mathbb{E}(X^r) = \frac{1}{n}\sum_{i=1}^{n} X_i^r.$$

Thus, for $r = 1, 2, 3$ in (4.2) and solving by numerical methods these equations, we can obtain the MM estimates. The `dfsane` function, `BFGS` method and `BB` package were used in the software R Core Team (Team et al., 2013).

4.3 Second kind statistic for the TIW distribution

In this section, we present one of our results for TIW distribution based on the Mellin Transform. The next, we will review some results from literature and then present the new results for TIW distribution.

4.3.1 Mellin Transform

The Mellin Transform relates to Laplace and Fourier transforms, and has been applied in different engineering fields (Bertrand et al., 1995). Let X be a random variable and $F(x)$ a function defined in real line to $[0, 1]$. The first characteristic function of the second kind is defined as the MT, denoted by $\phi_X(s)$ (Epstein, 1948; Nicolas, 2002)

$$\phi_X(s) = \int_0^\infty x^{s-1} dF(x) = \mathbb{E}\left(X^{s-1}\right). \tag{4.5}$$

Generally, the above integral does exist only for $s = a + bj$, with $a, b \in \mathbb{R}$ and j the imaginary number.

The second characteristic function of the second kind, denoted by $\varphi_X(s)$, is defined as the natural logarithm of (4.5)

$$\varphi_X(s) = \log\left(\phi_X(s)\right). \tag{4.6}$$

With this we motivation to formulate the next results.

Theorem 4.3.1. *Let X be a random variable following TIW distribution, $X \sim TIW(x; \beta, \eta, \lambda)$. The MT of X*

is given by

$$\phi_X(s) = \eta^{-\frac{s-1}{\beta}}\Gamma\left(1-\frac{s-1}{\beta}\right)\left(1+\lambda-\lambda 2^{\frac{s-1}{\beta}}\right). \tag{4.7}$$

The proof of Theorem 4.3.1 follows immediately from (4.2) and (4.5).

4.3.2 Log-Cumulants

There is a closely related between Log-Moments (LM) and LC. Now, we show LM are derived from MT similarly that the moments are obtained from the characteristic function.

Nicolas (2002) defines $\forall r \in \mathbb{N}$, the r-th LM or second-kind moment of the MT as following

$$\tilde{\mu}_r = \frac{d^r \phi_X}{ds^r}(s)\bigg|_{s=1} = \int_0^{\infty} (\log x)^r dF(x) = \mathbb{E}\left((\log X)^r\right).$$

The r-th LC is obtained from derivative of (4.6) and then by evaluating the function at $s=1$,

$$\tilde{\kappa}_r = \frac{d^r \varphi_X(s)}{ds^r}\bigg|_{s=1}. \tag{4.8}$$

Thus, the expressions for the first three cases and based on Anfinsen and Eltoft (2011) the r-th LC are given by

$$\begin{aligned}
\tilde{\kappa}_1 &= \tilde{\mu}_1, \\
\tilde{\kappa}_2 &= \tilde{\mu}_2 - \tilde{\mu}_1^2, \\
\tilde{\kappa}_3 &= \tilde{\mu}_3 - 3\tilde{\mu}_1\tilde{\mu}_2 + 2\tilde{\mu}_1^3, \\
&\vdots \\
\tilde{\kappa}_r &= \tilde{\mu}_r - \sum_{i=1}^{r-1}\binom{r-1}{i-1}\tilde{\kappa}_i\tilde{\mu}_{r-i},
\end{aligned} \tag{4.9}$$

where $\tilde{\mu}_r$ can be replace by (Nicolas, 2006)

$$\hat{\tilde{\mu}}_r = \frac{1}{n}\sum_{i=1}^{n}(\log x_i)^r, \tag{4.10}$$

here, n is the sample size and x_i indicates the i-th sample observation.

In Table 4.1, for $X \sim TIW(x;\beta,\eta,\lambda)$ we present the first six theoretical LC, where the polygamma function of order n, $\Psi^{(n)}(x) = \frac{d^{n+1}}{dx^{n+1}}\log\Gamma(x)$ is the $(n+1)$-th derivative of the logarithm of the gamma function.

Thus, the $\varphi_X(s)$ function for TIW distribution can be written as

$$\varphi_X(s) = (1-s)\frac{1}{\beta}\log(\eta) + \log\left(\Gamma\left(1-\frac{s-1}{\beta}\right)\right) + \log\left(1+\lambda-\lambda 2^{\frac{s-1}{\beta}}\right). \quad (4.11)$$

LC	$-1 < \lambda < 1$
$\tilde{\kappa}_1$	$-\frac{1}{\beta}\left[\log(\eta) + \lambda\log(2) + \Psi^{(0)}(1)\right]$
$\tilde{\kappa}_2$	$-\frac{1}{\beta^2}\left[\lambda\log^2(2)(\lambda+1) - \Psi^{(1)}(1)\right]$
$\tilde{\kappa}_3$	$-\frac{1}{\beta^3}\left[\lambda\log^3(2)\left(2\lambda^2+3\lambda+1\right) + \Psi^{(2)}(1)\right]$
$\tilde{\kappa}_4$	$-\frac{1}{\beta^4}\left[\lambda\log^4(2)\left(6\lambda^3+12\lambda^2+7\lambda+1\right) - \Psi^{(3)}(1)\right]$
$\tilde{\kappa}_5$	$-\frac{1}{\beta^5}\left[\lambda\log^5(2)\left(24\lambda^4+60\lambda^3+50\lambda^2+15\lambda+1\right) + \Psi^{(4)}(1)\right]$
$\tilde{\kappa}_6$	$-\frac{1}{\beta^6}\left[\lambda\log^6(2)\left(120\lambda^5+360\lambda^4+390\lambda^3+180\lambda^2+31\lambda+1\right) - \Psi^{(5)}(1)\right]$

Table 4.1: The first six Log-Cumulants of the TIW distribution

4.4 Estimation and GoF for the TIW distribution

4.4.1 Method of Log-Cumulants

In a similar way to Subsection 4.2.2, the proposed estimation method consists in equaling sample version LC in (4.10) with theoretical LC according to (4.9) as follow

$$\begin{aligned}
-\frac{1}{\hat{\beta}}\left[\log(\hat{\eta}) + \hat{\lambda}\log(2) + \Psi^{(0)}(1)\right] &= \hat{\tilde{\mu}}_1, \\
-\frac{1}{\hat{\beta}^2}\left[\hat{\lambda}\log^2(2)(\hat{\lambda}+1) - \Psi^{(1)}(1)\right] &= \hat{\tilde{\mu}}_2 - \hat{\tilde{\mu}}_1^2, \\
-\frac{1}{\hat{\beta}^3}\left[\hat{\lambda}\log^3(2)\left(2\hat{\lambda}^2+3\hat{\lambda}+1\right) + \Psi^{(2)}(1)\right] &= \hat{\tilde{\mu}}_3 - 3\hat{\tilde{\mu}}_1\hat{\tilde{\mu}}_2 + 2\hat{\tilde{\mu}}_1^3,
\end{aligned}$$

such that

$$\begin{aligned}
\hat{\beta} &= \sqrt[3]{\frac{\hat{\lambda}\left(2\hat{\lambda}^2+3\hat{\lambda}+1\right)\log^3(2) + \Psi^{(2)}(1)}{-\hat{\tilde{\mu}}_3 + 3\hat{\tilde{\mu}}_1\hat{\tilde{\mu}}_2 - 2\hat{\tilde{\mu}}_1^3}}, \\
\hat{\eta} &= \mathrm{e}^{-\hat{\beta}\hat{\tilde{\mu}}_1 - \hat{\lambda}\log(2) - \Psi^{(0)}(1)}, \\
\hat{\lambda} &= \frac{\sqrt{4\left[-\hat{\beta}^2\left(\hat{\tilde{\mu}}_2 - \hat{\tilde{\mu}}_1^2\right) - \Psi^{(1)}(1)\right]} - \sqrt{\log^2(2)}}{2\sqrt{\log^2(2)}}.
\end{aligned}$$

Remark 4.4.1. *An advantage of this method is that there is an expression in closed form. This does not happen* *in the above methods. Thus, the system is solved by non-linear optimization methods.*

This study was carried out by using the `BB` and `MaxLik` packages (Varadhan and Gilbert, 2010) and (Henningsen and Toomet, 2011), also by using `BBsolve`, `maxBFGS` functions avaliables in the software R Core Team.

4.4.2 The Log-Cumulant Diagram

Let us start this section mention that the Pearson system of distributions was presented by Delignon et al. (1997). It helps in the model selection and choosing the best fit for the data based on kurtosis and skewness measures. Nevertheless, Nicolas (2002) showed that the Pearson diagram sometimes could be complex in treating positive random variables and introduced the $(\tilde{\kappa}_3, \tilde{\kappa}_2)$ diagram, which uses the second statistics $\tilde{\kappa}_3$ and $\tilde{\kappa}_2$ rather than kurtosis and skewness measures.

There are different representations into the diagram, which is due to the number of parameters contained in LC expressions (Anfinsen and Eltoft, 2011). Hence, when there is no parameter in LC expressions, there is a zero-dimensional space; one parameter in LC is represented by a curve and a surface for two parameters.

Figure 4.2 displays the 2-dimensional space, where each dimension represents the third-order LC versus the second-order LC, referring to the TIW distribution. Here, we can plot the region obtained from the theoretical LC estimated. In the application section 4.6, the points representing the sample LC will be computed using a bootstrap method from data samples and graphed over the (κ_3, κ_2) diagram.

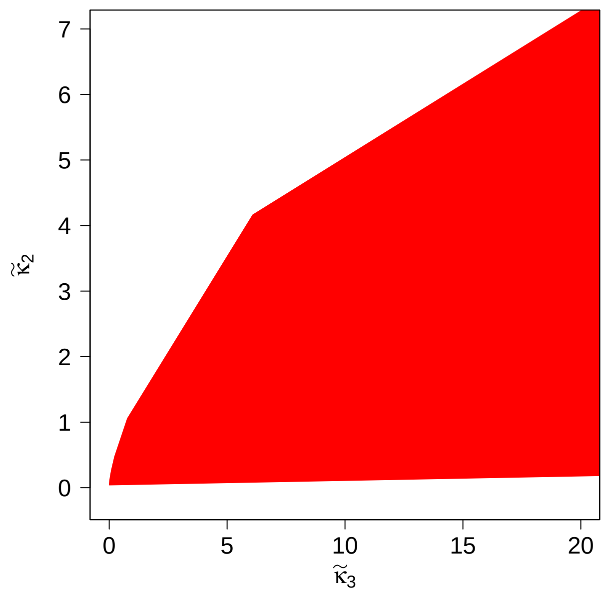


Figure 4.2: Diagram of Log-Cumulants showing the dimensional space of theoretical LC for the TIW distribution

4.4.3 Hotelling's Statistic

By the multivariate central limit theorem we know that for larger samples sizes, $\sqrt{\mathbf{n}}\left(\bar{\mathbf{X}}-\mu\right) \overset{\text{approx}}{\sim} \mathbb{N}\left(\mathbf{0},\boldsymbol{\Sigma}\right)$, and the covariance matrix follows a chi-square distribution with ν degrees of freedom, denoted by $\boldsymbol{\Sigma} = \mathbf{n}\left(\bar{\mathbf{X}}-\mu\right)^{\top}\mathbf{S^{-1}}\left(\bar{\mathbf{X}}-\mu\right) \overset{\text{approx}}{\sim} \chi^{\mathbf{2}}_{\nu}$, where μ is the mean vector, $\bar{\mathbf{X}}$ is the sample mean vector and $\mathbf{S}$ is the sample covariance matrix.

Now, the t-student distribution is used for small samples sizes. We can write $t = \frac{\bar{X}-\mu}{s/\sqrt{n}}$, as $T = \sqrt{n}\left(\bar{\mathbf{X}}-\mu\right)\mathbf{S^{-1}}$. If we take square to both terms, we obtain

$$T^2 = n\left(\bar{\mathbf{X}}-\mu\right)^{\top}\mathbf{S^{-1}}\left(\bar{\mathbf{X}}-\mu\right). \tag{4.12}$$

In this case, (4.12) will be Hotelling's T^2 statistic (Anderson, 1962; Hotelling, 1992). T^2 follows the F-Snedecor distribution with $\nu, n-\nu$ degrees of freedom denoted by $T^2 \sim F_{\nu,n-\nu}$.

Let us consider the next hypothesis testing:

$$H_0 := \mathbb{E}\left(\bar{\mathbf{X}}\right) = \mu \quad \text{vs} \quad H_1 := \mathbb{E}\left(\bar{\mathbf{X}}\right) \neq \mu. \tag{4.13}$$

Assume the null hypothesis H_0 true and a significance level α. The likelihood ratio test does not reject H_0, will be $T^2 \leq Q(1-\alpha; \nu, n-\nu)$, where $Q(\bullet; \nu, n-\nu)$ is the quantile function for $F_{\nu;n-\nu}$ (Anderson, 1962).

We aim to find the Hotelling's T^2 statistic to provide GoF tests using LC. Thus, we pretend to estimate the LC to identify the underlying distribution according to the location of the LC estimated $\begin{bmatrix} \hat{\tilde{\kappa}}_2 & \hat{\tilde{\kappa}}_3 \end{bmatrix}$ over the LC diagram (Vasconcelos et al., 2021).

An equivalent formulation of (4.13) will be needed

$$H_0 := \mathbb{E}\left(\begin{bmatrix} \hat{\tilde{\kappa}}_2 & \hat{\tilde{\kappa}}_3 \end{bmatrix}\right) = \begin{bmatrix} \tilde{\kappa}_2 & \tilde{\kappa}_3 \end{bmatrix} \quad \text{vs} \quad H_1 := \mathbb{E}\left(\begin{bmatrix} \hat{\tilde{\kappa}}_2 & \hat{\tilde{\kappa}}_3 \end{bmatrix}\right) \neq \begin{bmatrix} \tilde{\kappa}_2 & \tilde{\kappa}_3 \end{bmatrix}.$$

To reject or not the null hypothesis depends on the belonging of the LC estimated over the specific regions into the LC diagram (Anfinsen et al., 2011).

Thus, T^2 converges in distribution to a random variable following a chi-squared distribution with ν degrees of freedom

$$T^2 = n\left(\begin{bmatrix} \hat{\tilde{\kappa}}_2 \\ \hat{\tilde{\kappa}}_3 \end{bmatrix} - \begin{bmatrix} \tilde{\kappa}_2 \\ \tilde{\kappa}_3 \end{bmatrix}\right)^\top \widehat{\mathbf{K}}^{-1}\left(\begin{bmatrix} \hat{\tilde{\kappa}}_2 \\ \hat{\tilde{\kappa}}_3 \end{bmatrix} - \begin{bmatrix} \tilde{\kappa}_2 \\ \tilde{\kappa}_3 \end{bmatrix}\right) \xrightarrow{D} \chi^2_{2,\nu},$$

where $\widehat{\mathbf{K}}$ is the asymptotic covariance matrix and $\mathbb{P}\left(T^2 \leq \chi^2_{2,\nu}\right) = 1-\nu$.

Proposition 4.4.1. *Let X be a random variable following TIW distribution, $X \sim TIW(x; \beta, \eta, \lambda)$. The T^2 statistic based on LC Estimated is given by*

$$T^2 = \frac{n\beta^6}{\hat{\tau}_{33}\hat{\tau}_{22} - \hat{\tau}_{23}^2}\left[\hat{\tau}_{33}\left(\hat{\tilde{\kappa}}_2 - \tilde{\kappa}_2\right)^2 + \hat{\tau}_{22}\left(\hat{\tilde{\kappa}}_3 - \tilde{\kappa}_3\right)^2 - 2\hat{\tau}_{23}\left(\hat{\tilde{\kappa}}_2 - \tilde{\kappa}_2\right)\left(\hat{\tilde{\kappa}}_3 - \tilde{\kappa}_3\right)\right].$$

The proof of Proposition 4.4.1 and the algorithm to calculate T^2 is in Appendix, Section 4.8.

4.4.4 Ellipse Confidences

As a similar way than subsection 4.4.3, if μ is the mean of the normal multivariate distribution, that is, $\mathbb{N}(\mu, \mathbf{\Sigma})$, the probability is $1-\alpha$ of drawing a sample of the population with mean $\bar{X}$ and covariance matrix $\mathbf{S}$ such that

$$n\left(\bar{\mathbf{X}} - \mu\right)^\top \mathbf{S}^{-1}\left(\bar{\mathbf{X}} - \mu\right) \leq \mathbf{T}^2(\alpha). \tag{4.14}$$

Thus, if we compute 4.14 for a particular sample, we have confidence $1-\alpha$ that 4.14 is a true statement concerning μ (Anderson, 1962).

The inequality

$$n\left(\begin{bmatrix}\hat{\tilde{\kappa}}_2\\ \hat{\tilde{\kappa}}_3\end{bmatrix}-\begin{bmatrix}\tilde{\kappa}_2\\ \tilde{\kappa}_3\end{bmatrix}\right)^\top \widehat{\mathbf{K}}^{-1}\left(\begin{bmatrix}\hat{\tilde{\kappa}}_2\\ \hat{\tilde{\kappa}}_3\end{bmatrix}-\begin{bmatrix}\tilde{\kappa}_2\\ \tilde{\kappa}_3\end{bmatrix}\right)\leq T^2(\alpha),$$

is the interior and boundary of an ellipsoid of $[\tilde{\kappa}_2,\tilde{\kappa}_3]^\top$, with center at $\left[\hat{\tilde{\kappa}}_2,\hat{\tilde{\kappa}}_3\right]^\top$ and with size and shape depending on $\widehat{\mathbf{K}}^{-1}$ and α.

4.5 Simulation Study

We present a brief simulation study to verify the performance of the estimators. We use the ML, MM and LC methods for parameter estimation. Some scenarios with different sample sizes are shown in Table 4.2.

n	(β, η, λ)		MM			ML			LC		
			$\widehat{\beta}$	$\widehat{\eta}$	$\widehat{\lambda}$	$\widehat{\beta}$	$\widehat{\eta}$	$\widehat{\lambda}$	$\widehat{\beta}$	$\widehat{\eta}$	$\widehat{\lambda}$
30	$(2, 2, 0.9)$	Mean	2.903	4.306	0.818	2.509	3.803	0.772	2.414	3.554	0.882
		Bias	0.903	2.306	-0.082	0.509	1.803	-0.128	0.414	1.554	-0.018
		MSE	1.662	48.876	0.268	0.975	40.529	0.132	1.111	14.901	0.131
50			2.590	2.880	0.895	2.236	2.747	0.755	2.163	2.997	0.810
			0.590	0.880	-0.005	0.236	0.747	-0.145	0.163	0.997	-0.090
			0.807	4.865	0.194	0.277	3.118	0.138	0.468	9.312	0.165
80			2.498	2.576	0.926	2.179	2.582	0.767	2.104	2.731	0.812
			0.498	0.576	0.026	0.179	0.582	-0.133	0.104	0.731	-0.088
			0.625	2.996	0.183	0.188	2.062	0.129	0.334	6.250	0.166
100			2.409	2.348	0.965	2.139	2.490	0.768	2.097	2.707	0.779
			0.409	0.348	0.065	0.139	0.490	-0.132	0.097	0.707	-0.121
			0.516	2.049	0.171	0.129	1.535	0.127	0.280	5.275	0.180
30	$(9, 3.2, -0.9)$		10.662	2.304	-0.281	9.303	1.952	-0.046	9.492	1.960	-0.057
			1.662	-0.896	0.619	0.303	-1.248	0.854	0.492	-1.240	0.843
			12.729	2.206	1.029	8.271	2.230	1.379	10.572	2.426	1.313
50			9.908	2.668	-0.551	9.079	2.542	-0.481	9.286	2.538	-0.473
			0.908	-0.532	0.349	0.079	-0.658	0.419	0.286	-0.662	0.427
			4.538	1.260	0.506	2.256	1.026	0.568	3.738	1.126	0.541
80			9.703	2.836	-0.653	9.095	2.723	-0.601	9.246	2.746	-0.602
			0.703	-0.364	0.247	0.095	-0.477	0.299	0.246	-0.454	0.298
			3.143	1.067	0.365	1.360	0.703	0.347	2.518	0.826	0.337
100			9.485	2.981	-0.742	9.110	2.895	-0.713	9.174	2.960	-0.732
			0.485	-0.219	0.158	0.110	-0.305	0.187	0.174	-0.240	0.168
			1.897	0.898	0.250	0.663	0.437	0.170	1.398	0.593	0.178
30	$(3, 2, -0.7)$		4.334	2.209	-0.175	3.447	1.988	-0.573	3.812	2.088	-0.619
			1.334	0.209	0.525	0.447	-0.012	0.127	0.812	0.088	0.081
			4.624	15.403	4.930	1.226	0.739	0.199	2.186	1.277	0.113
50			3.722	2.159	-0.050	3.080	2.106	-0.720	3.264	2.134	-0.717
			0.722	0.159	0.650	0.080	0.106	-0.020	0.264	0.134	-0.017
			1.935	3.172	7.380	0.231	0.294	0.096	0.559	0.627	0.120
80			3.592	2.118	-0.129	3.016	2.133	-0.756	3.157	2.133	-0.736
			0.592	0.118	0.571	0.016	0.133	-0.056	0.157	0.133	-0.036
			1.442	2.989	3.730	0.119	0.222	0.072	0.324	0.494	0.113
100			3.497	2.085	-0.252	2.980	2.145	-0.769	3.083	2.134	-0.743
			0.497	0.085	0.448	-0.020	0.145	-0.069	0.083	0.134	-0.043
			1.073	2.180	2.396	0.060	0.173	0.059	0.175	0.392	0.103
30	$(1.5, 0.5, 0.04)$		1.499	1.310	0.613	1.655	0.462	0.212	2.102	0.519	0.383
			-0.001	0.810	0.573	0.155	-0.038	0.172	0.602	0.019	0.343
			0.001	1.504	41.762	0.301	0.032	0.251	0.947	0.064	0.278
50			1.494	0.939	0.730	1.465	0.473	0.253	1.730	0.552	0.282
			-0.006	0.439	0.690	-0.035	-0.027	0.213	0.230	0.052	0.242
			0.000	0.546	55.592	0.082	0.017	0.264	0.268	0.026	0.269
80			1.494	0.795	0.670	1.425	0.473	0.269	1.641	0.565	0.247
			-0.006	0.295	0.630	-0.075	-0.027	0.229	0.141	0.065	0.207
			0.000	0.246	19.319	0.058	0.013	0.259	0.162	0.021	0.262
100			1.495	0.682	0.812	1.407	0.474	0.274	1.568	0.569	0.184
			-0.005	0.182	0.772	-0.093	-0.026	0.234	0.068	0.069	0.144
			0.000	0.087	31.224	0.046	0.010	0.250	0.095	0.017	0.253

Table 4.2: Parameter estimates for TIW distribution in some scenarios

A simple simulation study with 1000 Monte Carlo experiments will be discussed to evaluate the performance of the MM, ML, and LC estimates. With different sample sizes such as 30, 50, 80 and 100. In all scenarios, we can observe good results associated with small values; small bias and mean squared error (MSE) values are associated with 80 and 100 sample sizes. We can observe that while the sample size increases, the bias and MSE decrease. However, for the last scenario, MM allows us to see that for λ values close to zero in the third parameter, the MSE is higher. This indicates that it is not a good estimate.

4.6 Application to real datasets

In this section, we provide an analysis of real datasets to evaluate the T^2 statistic. In the present study, we will be using the following three datasets:

- DATASET 1: The data are taken from Aarset (1987) and also reported in Khan and King (2014), which refers to the failure times of fifty devices put on life tests at time zero. This dataset is known to have a bathtub-shaped hazard rate. The observations (in weeks) are

0.1	0.2	1.0	1.0	1.0	1.0	1.0	2.0	3.0	6.0
7.0	11.0	12.0	18.0	18.0	18.0	18.0	18.0	21.0	32.0
36.0	40.0	45.0	45.0	47.0	50.0	55.0	60.0	63.0	63.0
67.0	67.0	67.0	67.0	72.0	75.0	79.0	82.0	82.0	83.0
84.0	84.0	84.0	85.0	85.0	85.0	85.0	85.0	86.0	86.0

Table 4.3: DATASET 1. Life time of 50 devices

- DATASET 2: The data in Lee and Wang (2003) represents a set of reported remission times (in months) of 128 bladder cancer patients.

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63	0.20	2.23	3.52	4.98	6.97
9.02	13.29	0.40	2.26	3.57	5.06	7.09	9.22	13.80	25.74	0.50	2.46	3.64
5.09	7.26	9.47	14.24	25.82	0.51	2.54	3.70	5.17	7.28	9.74	14.76	26.31
0.81	2.62	3.82	5.32	7.32	10.06	14.77	32.15	2.64	3.88	5.32	7.39	10.34
14.83	34.26	0.90	2.69	4.18	5.34	7.59	10.66	15.96	36.66	1.05	2.69	4.23
5.41	7.62	10.75	16.62	43.01	1.19	2.75	4.26	5.41	7.63	17.12	46.12	1.26
2.83	4.33	5.49	7.66	11.25	17.14	79.05	1.35	2.87	5.62	7.87	11.64	17.36
1.40	3.02	4.34	5.71	7.93	11.79	18.10	1.46	4.40	5.85	8.26	11.98	19.13
1.76	3.25	4.50	6.25	8.37	12.02	2.02	13.31	4.51	6.54	8.53	12.03	20.28
2.02	3.36	6.76	12.07	21.73	2.07	3.36	6.93	8.65	12.63	22.69		

Table 4.4: DATASET 2. Remission times of 128 bladder cancer patients

- DATASET 3: The data are the time between failures (thousands of hours) of 23 secondary reactor pumps installed in the RSG-GAS reactor in Suprawhardana and Prayoto (1999) and also Bebbington et al. (2007).

2.160	0.746	0.402	0.954	0.491	6.560	4.992	0.347	0.150
0.358	0.101	1.359	3.465	1.060	0.614	1.921	4.082	0.199
0.605	0.273	0.070	0.062	5.320				

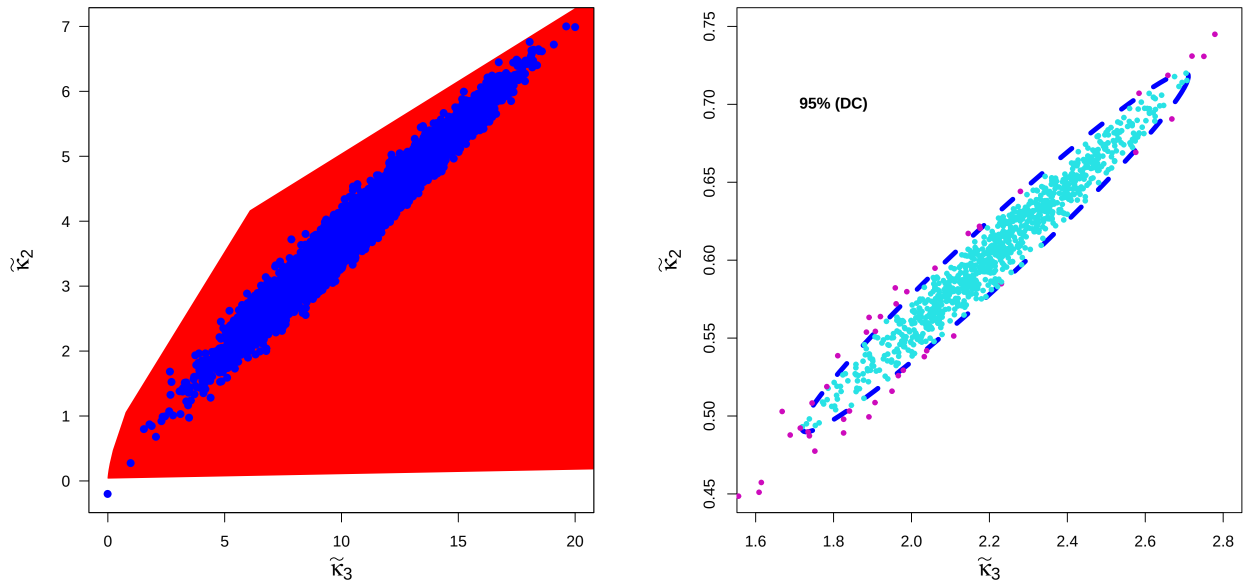
Table 4.5: DATASET 3. Time between failures of secondary reactor pumps

For the above datasets, in Table 4.6 we show some descriptive statistics for each one. Coefficient of Variation (CV) values indicate greater levels of dispersion in data around the mean. Related to asymmetry and kurtosis in dataset 1, they indicate left tail and smooth distribution. In dataset 2, for the mean, median, and mode values, we have skewed left distribution. Standard Deviation (SD) in dataset 3 observations indicate a low dispersed relation to the mean.

Values in Table 4.7 illustrates the T^2 statistic and p-value in each dataset. We can observe that in all estimation methods we verify the quality of our estimation methods and Goodness-of-Fit measures.

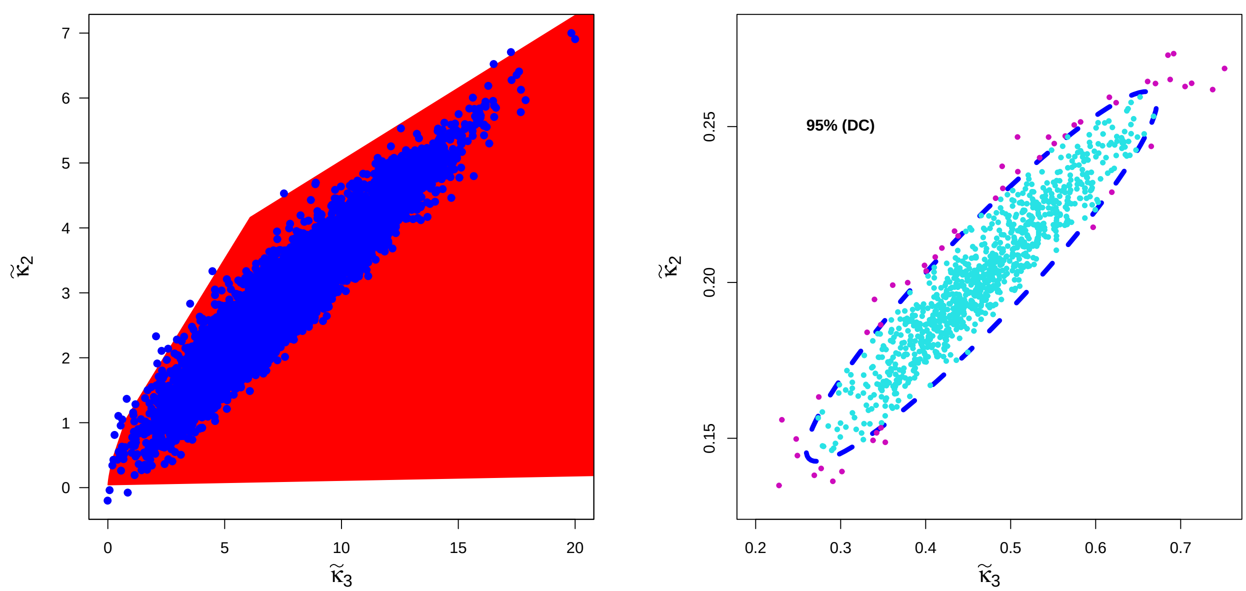
Dataset	**Min**	**Max**	**Mean**	**Median**	**Moda**	**SD**	**Asymmetry**	**Kurtosis**	**CV**
1	0.10	86.00	45.69	48.50	1.00	32.84	-0.13	-1.64	71.87
2	0.08	79.05	9.37	6.39	5.32	10.51	3.29	18.48	112.20
3	0.06	6.56	1.55	0.61	0.75	1.97	1.29	0.18	127.10

Table 4.6: Descriptive statistics for datasets



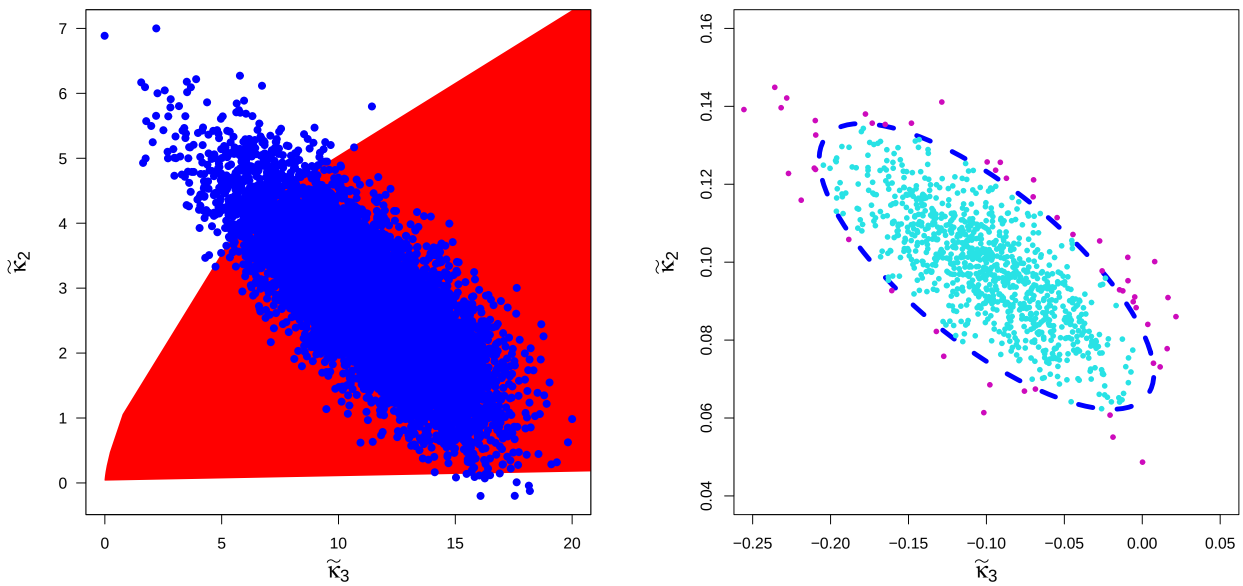
(a) (b)

Figure 4.3: Diagram of the LC $(\tilde{\kappa}_3, \tilde{\kappa}_2)$ and Confidence Ellipses for the dataset 1 for TIW distribution



(a) (b)

Figure 4.4: Diagram of the LC $(\tilde{\kappa}_3, \tilde{\kappa}_2)$ and Confidence Ellipses for the dataset 2 for TIW distribution



(a) (b)

Figure 4.5: Diagram of the LC $(\tilde{\kappa}_3, \tilde{\kappa}_2)$ and Confidence Ellipses for the dataset 3 for TIW distribution

Method	**Dataset 1**			**Dataset 2**			**Dataset 3**		
	Estimative $(\hat{\beta}, \hat{\eta}, \hat{\lambda})$	T^2	p-value	Estimative $(\hat{\beta}, \hat{\eta}, \hat{\lambda})$	T^2	p-value	Estimative $(\hat{\beta}, \hat{\eta}, \hat{\lambda})$	T^2	p-value
MM	0.750 2.194 -0.422	1.521	0.693	0.749 2.155 -0.477	8.196	0.049	1.487 1.647 -0.101	0.348	0.956
ML	0.512 0.518 -0.700	10.220	0.029	0.835 0.639 -0.855	4.092	0.263	0.847 3.262 -0.442	1.675	0.683
LC	2.468 0.001 -0.796	0.276	0.966	2.447 0.034 -0.533	0.181	0.980	1.429 4.698 -0.578	0.298	0.964

Table 4.7: Estimatives, T^2 statistic and p-value for the datasets.

As an additional analysis, we show a visual illustration form of fits with confidence ellipses for each dataset. We use the T^2 statistic value to measure the distance between the data and the TIW model. The better fit between the data and the model means a smaller T^2 statistic and a higher p-value do not reject the null hypothesis.

The ellipses was done by the ellipse-package available in the software R. The ellipse center will be the LC $\hat{\tilde{\kappa}}_2, \hat{\tilde{\kappa}}_3$ and the axes are directed according to the eigenvectors of $\widehat{\mathbf{K}}$; the Degree Coverage (DC) is quantified with a significance level of 95%. See Figures 4.3, 4.4 and 4.5.

For Dataset 1, the MM and LC methods have a p-value greater than the ML method, and also T^2 is remarkably higher than the last method. Figure 4.3 shows the majority of points into the diagram and into the ellipse. That means a good fit.

T^2 in Dataset 2 has a smaller value in the LC method than others, and seeing the p-value higher means the best fit, such that Dataset 3 have the same analysis.

Figure 4.4 shows most of the points on the diagram and on the ellipse. That means a good fit. However, in Figure 4.5, despite the points perpendicular to the diagram, they are on the diagram and most of them are on the ellipse.

4.7 Concluding remarks

We provide analitical and visual analysis to Goodness-of-Fit measures to the datasets. We define the Mellin Transform for Transmuted Inverse Weibull distribution and give the close form of the estimators. We estimate the parameters model using Maximum Likelihood, Moments and Log-Cumulants and show the better fit in the specific dataset by T^2 statistic. A simple Monte Carlo simulation study accomplished the performance of estimates.

4.8 Appendix Chapter

Algorithm to calculate T^2 statistic:	
Step 1	Estimate the β, η, λ parameters by ML, MM or LC methods,
Step 2	Take the estimatives from above step and find $\hat{\tilde{\kappa}}_2$ and $\hat{\tilde{\kappa}}_3$,
Step 3	Find sample LC: $\tilde{\kappa}_2$ and $\tilde{\kappa}_3$ through Log Moments,
Step 4	Calculate T^2 statistic with ML, MM or LC.

$$\begin{aligned}
\mathbf{K_{3x3}} &= \mathbf{J_{3x3}^{\top} M_{3x3} J_{3x3}} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ -2\mu_1 & 1 & 0 \\ -3(\mu_2 - 2\mu_1^2) & -3\mu_1 & 1 \end{pmatrix}^{\top} \begin{pmatrix} \tilde{\kappa}_2 & \tilde{\kappa}_3 + 2\tilde{\kappa}_1\tilde{\kappa}_2 & M_{1,3} \\ \tilde{\kappa}_3 + 2\tilde{\kappa}_1\tilde{\kappa}_2 & M_{2,2} & M_{2,3} \\ M_{3,1} & M_{3,2} & M_{3,3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2\mu_1 & 1 & 0 \\ -3(\mu_2 - 2\mu_1^2) & -3\mu_1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \tilde{\kappa}_2 & \tilde{\kappa}_3 & \tilde{\kappa}_4 \\ \tilde{\kappa}_3 & \tilde{\kappa}_4 + 2\tilde{\kappa}_2^2 & \tilde{\kappa}_5 + 6\tilde{\kappa}_2\tilde{\kappa}_3 \\ \tilde{\kappa}_4 & \tilde{\kappa}_5 + 6\tilde{\kappa}_2\tilde{\kappa}_3 & \tilde{\kappa}_6 + 9\tilde{\kappa}_2\tilde{\kappa}_4 + 9\tilde{\kappa}_3^2 + 6\tilde{\kappa}_2^3 \end{pmatrix},
\end{aligned}$$

where

$$
\begin{aligned}
M_{1,3} &= \tilde{\kappa}_4 + 3\tilde{\kappa}_1\tilde{\kappa}_3 + 3\tilde{\kappa}_2^2 + 3\tilde{\kappa}_1^2\tilde{\kappa}_2 = M_{3,1};\\
M_{2,2} &= \tilde{\kappa}_4 + 4\tilde{\kappa}_1\tilde{\kappa}_3 + 2\tilde{\kappa}_2^2 + 4\tilde{\kappa}_1^2\tilde{\kappa}_2;\\
M_{2,3} &= \tilde{\kappa}_5 + 5\tilde{\kappa}_1\tilde{\kappa}_4 + 9\tilde{\kappa}_2\tilde{\kappa}_3 + 9\tilde{\kappa}_1^2\tilde{\kappa}_3 + 12\tilde{\kappa}_1\tilde{\kappa}_2^2 + 6\tilde{\kappa}_1^3\tilde{\kappa}_2 = M_{3,2};\\
M_{3,3} &= \tilde{\kappa}_6 + 6\tilde{\kappa}_1\tilde{\kappa}_5 + 15\tilde{\kappa}_2\tilde{\kappa}_4 + 15\tilde{\kappa}_1^2\tilde{\kappa}_4 + 9\tilde{\kappa}_3^2 + 54\tilde{\kappa}_1\tilde{\kappa}_2\tilde{\kappa}_3 + 18\tilde{\kappa}_1^3\tilde{\kappa}_3 + 15\tilde{\kappa}_2^3 + 36\tilde{\kappa}_1^2\tilde{\kappa}_2^2 + 9\tilde{\kappa}_1^4\tilde{\kappa}_2.
\end{aligned}
$$

Based on (4.15), we obtain the asymptotic covariance matrix from LC ($\mathbf{K_{LC}}$) for TIW,

$$
\mathbf{K_{LC}} = \begin{pmatrix} \tilde{\kappa}_4 + 2\tilde{\kappa}_2^2 & \tilde{\kappa}_5 + 6\tilde{\kappa}_2\tilde{\kappa}_3 \\ \tilde{\kappa}_5 + 6\tilde{\kappa}_2\tilde{\kappa}_3 & \tilde{\kappa}_6 + 9\tilde{\kappa}_2\tilde{\kappa}_4 + 9\tilde{\kappa}_3^2 + 6\tilde{\kappa}_2^3 \end{pmatrix} = \frac{1}{\beta^6}\begin{pmatrix} \tau_{2,2} & \tau_{2,3} \\ \tau_{3,2} & \tau_{3,3} \end{pmatrix},
$$

where

$$
\begin{aligned}
\tau_{2,2} &= \beta^2\left\{-\left(\lambda\log^4(2)\left(6\lambda^3+12\lambda^2+7\lambda+1\right)+\Psi^{(3)}(1)\right)+2\left(-\left(\lambda\log^2(2)(\lambda+1)+\Psi^{(1)}(1)\right)\right)^2\right\},\\
\tau_{2,3} &= \beta\left\{-\left(\lambda\log^5(2)\left(24\lambda^4+60\lambda^3+50\lambda^2+15\lambda+1\right)+\Psi^{(4)}(1)\right)\right\}\\
&\quad +\beta\left\{6\left(-\left(\lambda\log^2(2)(\lambda+1)+\Psi^{(1)}(1)\right)\right)\left(-\left(\lambda\log^3(2)\left(2\lambda^2+3\lambda+1\right)+\Psi^{(2)}(1)\right)\right)\right\}\\
&= \tau_{3,2},\\
\tau_{3,3} &= -\left(\lambda\log^6(2)\left(120\lambda^5+360\lambda^4+390\lambda^3+180\lambda^2+31\lambda+1\right)+\Psi^{(5)}(1)\right)\\
&\quad +9\left(-\left(\lambda\log^2(2)(\lambda+1)+\Psi^{(1)}(1)\right)\right)\left(-\left(\lambda\log^4(2)\left(6\lambda^3+12\lambda^2+7\lambda+1\right)+\Psi^{(3)}(1)\right)\right)\\
&\quad +9\left(-\left(\lambda\log^3(2)\left(2\lambda^2+3\lambda+1\right)+\Psi^{(2)}(1)\right)\right)^2+6\left(-\left(\lambda\log^2(2)(\lambda+1)+\Psi^{(1)}(1)\right)\right)^3.
\end{aligned}
$$

If $\mathbf{K_{LC}}$ is nonsingular, then

$$
\mathbf{K_{LC}^{-1}} = \frac{\beta^6}{\tau_{33}\tau_{22}-\tau_{23}^2}\begin{pmatrix} \tau_{33} & -\tau_{32} \\ -\tau_{23} & \tau_{22} \end{pmatrix}.
$$

Finally, T^2 statistic is given by

$$
T^2 = n\left(\begin{bmatrix}\hat{\tilde{\kappa}}_2 \\ \hat{\tilde{\kappa}}_3\end{bmatrix} - \begin{bmatrix}\tilde{\kappa}_2 \\ \tilde{\kappa}_3\end{bmatrix}\right)^\top \widehat{\mathbf{K}}_{LC}^{-1}\left(\begin{bmatrix}\hat{\tilde{\kappa}}_2 \\ \hat{\tilde{\kappa}}_3\end{bmatrix} - \begin{bmatrix}\tilde{\kappa}_2 \\ \tilde{\kappa}_3\end{bmatrix}\right). \tag{4.15}
$$

Further manipulation on the above equation leads to obtain

$$T^2 = \frac{n\beta^6}{\hat{\tau}_{3,3}\hat{\tau}_{2,2} - \hat{\tau}_{2,3}^2}\left[\hat{\tau}_{3,3}\left(\hat{\tilde{\kappa}}_2 - \tilde{\kappa}_2\right)^2 + \hat{\tau}_{2,2}\left(\hat{\tilde{\kappa}}_3 - \tilde{\kappa}_3\right)^2 - 2\hat{\tau}_{2,3}\left(\hat{\tilde{\kappa}}_2 - \tilde{\kappa}_2\right)\left(\hat{\tilde{\kappa}}_3 - \tilde{\kappa}_3\right)\right] \leq \chi^2_{2,\nu}.$$

Other way to obtain the asymptotic covariance matrix from ML ($\mathbf{K_{ML}}$) is by using the observed information matrix instead $\mathbf{M_{3x3}}$ as following

$$\mathbf{K_{ML}} = \mathbf{Z}^\top \mathbf{H} \mathbf{Z},$$

where $\mathbf{H} = -\frac{\partial^2 l(\beta,\eta,\lambda)}{\partial(\beta,\eta,\lambda)^\top \partial(\beta,\eta,\lambda)} = \begin{pmatrix} \mathbf{H}_{\beta\beta} & \mathbf{H}_{\beta\eta} & \mathbf{H}_{\beta\lambda} \\ \mathbf{H}_{\eta\beta} & \mathbf{H}_{\eta\eta} & \mathbf{H}_{\eta\lambda} \\ \mathbf{H}_{\lambda\beta} & \mathbf{H}_{\lambda\eta} & \mathbf{H}_{\lambda\lambda} \end{pmatrix}$, is the Hessian matrix from 4.4 and

$$\mathbf{Z} = \begin{pmatrix} \frac{\partial}{\partial\beta}\tilde{\kappa}_2 & \frac{\partial}{\partial\beta}\tilde{\kappa}_3 \\ \frac{\partial}{\partial\eta}\tilde{\kappa}_2 & \frac{\partial}{\partial\eta}\tilde{\kappa}_3 \\ \frac{\partial}{\partial\lambda}\tilde{\kappa}_2 & \frac{\partial}{\partial\lambda}\tilde{\kappa}_3 \end{pmatrix} = \begin{pmatrix} \frac{2}{\beta^3}\left(\lambda \log^2(2)(\lambda+1) + \Psi^{(1)}(1)\right) & \frac{3\left(\lambda(2\lambda^2+3\lambda+1)\log^3(2) + \Psi^{(2)}(1)\right)}{\beta^4} \\ 0 & 0 \\ -\frac{(2\lambda+1)\log^2(2)}{\beta^2} & -\frac{(6\lambda^2+6\lambda+1)\log^3(2)}{\beta^3} \end{pmatrix}.$$

Thus,

$$\mathbf{K_{ML}} = \frac{\mathbf{H}_{\beta\beta}\mathbf{H}_{\lambda\lambda} - \mathbf{H}^{\mathbf{2}}_{\eta\lambda}}{\beta^{\mathbf{8}}|\mathbf{L}|}\begin{pmatrix} 4\beta^2\Psi^{(1)}(1)^2 & 6\beta^2\Psi^{(1)}(1)\Psi^{(2)}(1) \\ 6\beta\Psi^{(1)}(1)\Psi^{(2)}(1) & 9\Psi^{(2)}(1)^2 \end{pmatrix},$$

$$|\mathbf{L}| = \mathbf{H}_{\beta\beta}(\mathbf{H}_{\eta\eta}\mathbf{H}_{\lambda\lambda} - \mathbf{H}^{\mathbf{2}}_{\eta\lambda}) + \mathbf{H}_{\beta\eta}(\mathbf{H}_{\beta\lambda}\mathbf{H}_{\eta\lambda} - \mathbf{H}_{\beta\eta}\mathbf{H}_{\lambda\lambda}) + \mathbf{H}_{\beta\lambda}(\mathbf{H}_{\beta\eta}\mathbf{H}_{\eta\lambda} - \mathbf{H}_{\beta\lambda}\mathbf{H}_{\eta\eta}).$$

Finally, the T^2 statistic by using $\mathbf{K_{ML}}$ is

$$T^2 = n\left(\begin{bmatrix}\hat{\tilde{\kappa}}_2 \\ \hat{\tilde{\kappa}}_3\end{bmatrix} - \begin{bmatrix}\tilde{\kappa}_2 \\ \tilde{\kappa}_3\end{bmatrix}\right)^\top \widehat{\mathbf{K}}_{ML}^{-1}\left(\begin{bmatrix}\hat{\tilde{\kappa}}_2 \\ \hat{\tilde{\kappa}}_3\end{bmatrix} - \begin{bmatrix}\tilde{\kappa}_2 \\ \tilde{\kappa}_3\end{bmatrix}\right). \tag{4.16}$$

We furnish the expression expanded

$$T^2 = \frac{n\hat{\beta}^6}{4}\left(\frac{1}{\hat{\beta}^2} - \frac{1}{\beta^2}\right)^2\left(\frac{\left|\hat{\mathbf{L}}\right|}{\mathbf{H}_{\hat{\beta}\hat{\beta}}\mathbf{H}_{\hat{\lambda}\hat{\lambda}} - \mathbf{H}_{\hat{\eta}\hat{\lambda}}}\right).$$

Chapter 5

A Simulation Study for Transmuted Inverse Weibull Distribution

Abstract

A general simulation study from Transmuted Inverse Weibull distribution is shown in this article. This distribution generalizes some other distributions in the literature. Some properties of the Transmuted Inverse Weibull distribution are presented. We present a simulation study to verify the performance estimators with the Moments, Maximum Likelihood and Log-Cumulants methods. Kolmogorov Smirnov and Cramér-vom Mises as Goodness-of-Fit criterias are used. Monte Carlo methods such as bootstrap and Jackknife are uses to estimate population characteristics. Finally, an application to real dataset is provided.

Keywords: Bootstrap, Estimation Methods, Jackknife, Goodness-of-Fit, T^2 Statistic.

5.1 Introduction

The Transmuted Inverse Weibull (TIW) distribution can be a lifetime probability distribution that is used in differents fields. In this work, we have a simulation study with the three parameters TIW distribution.

The focus in this work is to evaluate the performance of the purpose estimators. We do Monte Carlo simulations with finite random samples from TIW. Also, with different scenarios and sample sizes we use the Moments, Maximum Likelihood and Log-Cumulats methods to estimate the model parameters.

In addition to verify the power of the GoF measures, we work Bootstrap and Jacknife methods for check the quality of the tools covered in this material; these methods are a class of Monte Carlo methods that estimate the distribution of a population by resampling.

This Chapter is organized as follows: In Section 5.2, the cdf, pdf, quantile function, mean and median of TIW distribution are presented. Next, in Section 5.3 the parameter estimation methods are displayed. In Section 5.4 a simulation study with some scenarios we estimate the bias and MSE for different methods, besides we estimate the power and Goodness-of-Fit measures for TIW distribution such as Cramér-von Mises (W), Kolmogorov Smirnov (KS) and we compare the T^2 measure. And finally, the conclusions an Future Works.

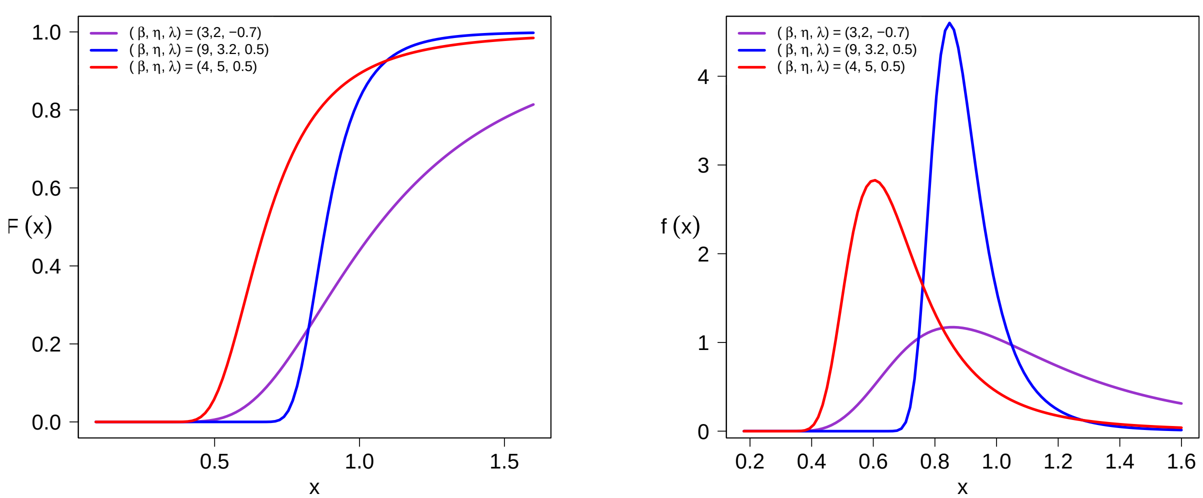
5.2 Transmuted Inverse Weibull

A random variable X is said to follow Transmuted Inverse Weibull distribution, with $\beta, \eta > 0$, and $|\lambda| \leq 1$, the shape, scale and transmuting parameters, respectively, if the CDF and PDF are given by (Khan and King, 2014):

$$F_X(x) = (1+\lambda)\, \mathrm{e}^{-\frac{1}{\eta}\left(\frac{1}{x}\right)^{\beta}} - \lambda \left(\mathrm{e}^{-\frac{1}{\eta}\left(\frac{1}{x}\right)^{\beta}}\right)^2,$$

$$f_X(x) = \left(\frac{\beta}{\eta}\right)\left(\frac{1}{x}\right)^{\beta+1} \mathrm{e}^{-\frac{1}{\eta}\left(\frac{1}{x}\right)^{\beta}} \left(1+\lambda-2\lambda\, \mathrm{e}^{-\frac{1}{\eta}\left(\frac{1}{x}\right)^{\beta}}\right), \quad x > 0. \tag{5.1}$$

Figure (5.1) shows the CDF and PDF for selected parameter values. Below are three PDF graphs, varying β, η and λ, the TIW tends to be a asymmetry distribution in all cases.



(a) (b)

Figure 5.1: Transmuted Inverse Weibull CDF (a) and PDF (b) for different values of β, η, λ

When $\lambda = 0$, TIW distribution simplifies to Inverse Weibull. When $\beta = 1$ and $\lambda = 0$, TIW distribution reduces to Transmuted Inverse Exponential. If $\beta = 2$, the Transmuted Inverse Rayleigh is a special case of TIW.

Using v a random number from zero to one. The quantile function from TIW, by solving $F_X(x) \leq v$ is

$$x = \left(-\eta \log\left(\frac{(1+\lambda) - \sqrt{(1+\lambda)^2 - 4\lambda v}}{2\lambda}\right)\right)^{\frac{-1}{\beta}}. \tag{5.2}$$

The r-th moment from $X \sim TIW(x;\beta,\eta,\lambda)$ distribution is given as follows

$$\mathbb{E}\left(X^r\right) = \eta^{\,-\frac{r}{\beta}}\Gamma\left(1 - \frac{r}{\beta}\right)\left(1 + \lambda - \lambda 2^{\frac{r}{\beta}}\right), \tag{5.3}$$

where $\Gamma(\cdot)$ is the Gamma function.

Using $v = 0.5$ in (5.2), we obtain the median of the TIW. In practice, this is the life at which at least 50% of the units will be expected to fail.

5.3 Parameter Estimation

5.3.1 Method of Moments

This method is based on the theoretical and sampling moments of the $X_1, X_2, \ldots, X_n$ random variables. For $r > 0$ integer, the r-th sample moment is the random variable

$$\mathbb{E}(X^r) = \frac{1}{n}\sum_{i=1}^{n} X_i^r. \tag{5.4}$$

The first three moments are:

$$\begin{aligned}
\mathbb{E}\left(X^1\right) &= \eta^{\,-\frac{1}{\beta}}\Gamma\left(1 - \frac{1}{\beta}\right)\left(1 + \lambda - \lambda 2^{\frac{1}{\beta}}\right) = \mu_1,\\
\mathbb{E}\left(X^2\right) &= \eta^{\,-\frac{2}{\beta}}\Gamma\left(1 - \frac{2}{\beta}\right)\left(1 + \lambda - \lambda 2^{\frac{2}{\beta}}\right) = \mu_2,\\
\mathbb{E}\left(X^3\right) &= \eta^{\,-\frac{3}{\beta}}\Gamma\left(1 - \frac{3}{\beta}\right)\left(1 + \lambda - \lambda 2^{\frac{3}{\beta}}\right) = \mu_3.
\end{aligned}$$

Solving by numerical methods (5.4), we can obtain the MM estimatives. The `dfsane` function, `BFGS` method and `BB` package were used in the software R.

- **Algorithm:**

  1. Let $x = (x_1, x_2, \ldots, x_n)$ a random sample following the TIW distribution;
  2. Write the simultaneus equations for $r = 1, 2, 3$,

$$\mathbb{E}(X^r) = \frac{1}{n}\sum_{i=1}^{n} X_i^r = \mu_r;$$

  3. Solve the equations by using the software R.

5.3.2 Method of Maximun Likelihood

Let $X_1, X_2, \ldots, X_n$ a random sample from TIW distribution with pdf in (5.1). The Maximum Likelihood function is given by

$$L(\beta,\eta,\lambda;x_i) = \prod_{i=1}^{n} \frac{\beta}{\eta}\left(\frac{1}{x_i}\right)^{\beta+1} \mathrm{e}^{-\frac{1}{\eta}\left(\frac{1}{x_i}\right)^{\beta}} \left(1+\lambda-2\lambda\,\mathrm{e}^{-\frac{1}{\eta}\left(\frac{1}{x_i}\right)^{\beta}}\right), \tag{5.5}$$

where $x_i, i = 1, 2, \ldots, n$ are the observed values of the random sample.

To estimate the parameters, we find the set of values of β, η and λ that attains their maximum in (5.5).

The log-likelihood function of (5.5) is as follows

$$l(\beta,\eta,\lambda;x_i) = n\log\frac{\beta}{\eta} + (\beta+1)\sum_{i=1}^{n}\log\left(\frac{1}{x_i}\right) - \sum_{i=1}^{n}\frac{1}{\eta}\left(\frac{1}{x_i}\right)^{\beta} + \sum_{i=1}^{n}\log\left(1+\lambda-2\lambda\,\mathrm{e}^{-\frac{1}{\eta}\left(\frac{1}{x_i}\right)^{\beta}}\right). \tag{5.6}$$

The Maximum Likelihood estimators are obtained by maximizing (5.6).

Thus, solving the below equations, the Maximum Likelihood Estimators can be obtained:

$$\begin{aligned}
\frac{\partial l}{\partial \beta}(\beta,\eta,\lambda;x_i) &= \frac{n}{\beta} + 2\sum_{i=1}^{n}\frac{\lambda\left(\frac{1}{x_i}\right)^{\beta}\log\left(\frac{1}{x_i}\right)\mathrm{e}^{-\left(\frac{1}{\eta}\right)\left(\frac{1}{x_i}\right)^{\beta}}}{\eta\left(\lambda-2\lambda\,\mathrm{e}^{-\frac{1}{\eta}\left(\frac{1}{x_i}\right)^{\beta}}+1\right)} - \frac{1}{\eta}\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)^{\beta}\log\left(\frac{1}{x_i}\right) + \sum_{i=1}^{n}\log\left(\frac{1}{x_i}\right) = 0,\\
\frac{\partial l}{\partial \eta}(\beta,\eta,\lambda;x_i) &= -\frac{n}{\eta} - 2\sum_{i=1}^{n}\frac{\lambda\left(\frac{1}{x_i}\right)^{\beta}\mathrm{e}^{-\left(\frac{1}{\eta}\right)\left(\frac{1}{x_i}\right)^{\beta}}}{\eta^2\left(\lambda-2\lambda\,\mathrm{e}^{-\frac{1}{\eta}\left(\frac{1}{x_i}\right)^{\beta}}+1\right)} + \frac{1}{\eta^2}\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)^{\beta} = 0,\\
\frac{\partial l}{\partial \lambda}(\beta,\eta,\lambda;x_i) &= \sum_{i=1}^{n}\frac{1-2\,\mathrm{e}^{-\left(\frac{1}{\eta}\right)\left(\frac{1}{x_i}\right)^{\beta}}}{\lambda-2\lambda\,\mathrm{e}^{-\left(\frac{1}{\eta}\right)\left(\frac{1}{x_i}\right)^{\beta}}+1} = 0.
\end{aligned}$$

In this work we use for maximization the `optim` function, `BFGS` method and `BB` package in the software R.

- **Algorithm:**

  1. Let $x = (x_1, x_2, \ldots, x_n)$ a random sample following the TIW distribution;
  2. Write the log-likelihood from TIW to get the parameters;
  3. Solve the equations by using the software R.

5.3.3 Method of Log-Cumulants

This method consists in equaling sample version LC with theoretical LC as follow

$$\begin{aligned} -\frac{1}{\hat{\beta}}\left(\log(\hat{\eta}) + \hat{\lambda}\log(2) + \Psi^{(0)}(1)\right) &= \hat{\tilde{\mu}}_1, \\ -\frac{1}{\hat{\beta}^2}\left(\hat{\lambda}\log^2(2)(\hat{\lambda}+1) - \Psi^{(1)}(1)\right) &= \hat{\tilde{\mu}}_2 - \hat{\tilde{\mu}}_1^2, \\ -\frac{1}{\hat{\beta}^3}\left(\hat{\lambda}\log^3(2)\left(2\hat{\lambda}^2 + 3\hat{\lambda} + 1\right) + \Psi^{(2)}(1)\right) &= \hat{\tilde{\mu}}_3 - 3\hat{\tilde{\mu}}_1\hat{\tilde{\mu}}_2 + 2\hat{\tilde{\mu}}_1^3, \end{aligned} \tag{5.7}$$

such that

$$\begin{aligned} \hat{\beta} &= \sqrt[3]{\frac{\hat{\lambda}\log^3(2)\left(2\hat{\lambda}^2 + 3\hat{\lambda} + 1\right) + \Psi^{(2)}(1)}{-\hat{\tilde{\mu}}_3 + 3\hat{\tilde{\mu}}_1\hat{\tilde{\mu}}_2 - 2\hat{\tilde{\mu}}_1^3}}, \\ \hat{\eta} &= e^{-\hat{\beta}\hat{\tilde{\mu}}_1 - \hat{\lambda}\log(2) - \Psi^{(0)}(1)}, \\ \hat{\lambda} &= \frac{\sqrt{4\left(-\hat{\beta}^2\left(\hat{\tilde{\mu}}_2 - \hat{\tilde{\mu}}_1^2\right) - \Psi^{(1)}(1)\right)} - \sqrt{\log^2(2)}}{2\sqrt{\log^2(2)}}. \end{aligned}$$

This study was done by using the `BB` and `MaxLik` packages, also by using `BBsolve`, `maxBFGS` functions avaliables in the software R.

- **Algorithm:**

  1. Let $x = (x_1, x_2, \ldots, x_n)$ a random sample following the TIW distribution;
  2. Write the simultaneus equations in 5.7;
  3. Solve the equations by using the software R.

5.3.4 Methods with resampling

Jackknife: The method is as following.

Let $X = (X_1, X_2, \ldots, X_n)$ a random sample of size n. Suppose that we are interested in estimating a parameter θ, and the estimate of θ computed from the sample is denoted as $\hat{\theta}$. The jackknife method creates jackknife samples $X_{(i)}$ that leave out one observation at a time, $X_{(i)} = (X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$ for $i = 1, 2, \ldots, n$, and so on to finally leave out the last observation to create the $n-$th jackknife sample, $X_{(n)} = (X_1, X_2, \ldots, X_{n-1})$. Then the jackknife method computes the value of $\hat{\theta}$ for each of the jackknife samples, $\hat{\theta}_{(i)}$, which is called the $i-$th jackknife replication, denotes the value of $\hat{\theta}$ computed from the $i-$th jackknife sample $X_{(i)}$ (Sinharay, 2010).

Bootstrap: The bootstrap is a computational method, which consists in a resampling method, based on calculating estimates from repeated sampling within the same sample, replacing analytical calculations with computational effort.

The bootstrap could be as following: Let $X_i = x_1, x_2, \ldots, x_n$ sample random with $i = 1, 2, \ldots, n$,

1) To calculate $\hat{\theta}$ estatistic of interest;

2) From the original sample extract, with replacement, a sample $X_i^{*s} = x_1^{*s}, x_2^{*s}, \ldots, x_n^{*s}$;

3) To calculate the same statistic of interest considering the bootstrap sample x^s to obtain $\hat{\theta}^{*s}$;

4) To repeat steps 2) and 3) a number S, very large number of times, obtaining $\hat{\theta}^{*1}, \hat{\theta}^{*2}, \ldots, \hat{\theta}^{*S}$;

5) Use the distribution estimate given by $\hat{\theta}^{*s}$ for $s = 1, 2, \ldots, S$, to obtain the statistic desired.

5.4 A Simulation Study

This section starts with a simulation study from TIW distribution. A Monte Carlo simulation study is carried out to compare the performance of the estimators. The simulation experiment was repeated 10.000 independent times and is sufficiently large to have stable results, each one with sample sizes $n = 30, 50, 80, 100$ and (β, η, λ) in the next scenarios $(3, 2, -0.7), (9, 3.2, 0.5), (4, 5, 0.5)$. We estimated the parameters using the method of MM, and then these estimates were used as the initial values in ML and LC. We calculate the mean, bias, and mean squared error estimates to verify the model's performance. Numerical results are complemented with graphics such as standard deviation and mean squared error to complement the analysis.

n	(β, η, λ)		MM			ML			LC		
			$\widehat{\beta}$	$\widehat{\eta}$	$\widehat{\lambda}$	$\widehat{\beta}$	$\widehat{\eta}$	$\widehat{\lambda}$	$\widehat{\beta}$	$\widehat{\eta}$	$\widehat{\lambda}$
30	$(3, 2, -0.7)$	Mean	3.055	1.937	-0.735	2.997	1.995	-0.753	3.028	1.945	-0.597
		Bias	0.055	-0.063	-0.035	-0.003	-0.005	-0.005	0.028	-0.055	0.103
		MSE	0.008	0.026	0.026	0.027	0.028	0.028	0.029	0.028	0.017
50			3.065	2.021	-0.756	2.993	1.998	-0.736	3.051	1.932	-0.599
			0.065	0.021	-0.056	-0.007	-0.002	-0.002	0.051	-0.068	0.101
			0.007	0.025	0.032	0.027	0.029	0.029	0.029	0.029	0.017
80			3.069	1.983	-0.733	2.992	2.001	-0.730	3.056	1.914	-0.603
			0.069	-0.017	-0.033	-0.008	0.001	0.001	0.056	-0.086	0.097
			0.009	0.024	0.026	0.025	0.027	0.027	0.029	0.030	0.017
100			3.057	2.003	-0.732	2.993	2.003	-0.722	3.055	1.909	-0.605
			0.057	0.003	-0.032	-0.007	0.003	0.003	0.055	-0.091	0.095
			0.008	0.024	0.024	0.025	0.027	0.027	0.028	0.031	0.017
30	$(9, 3.2, 0.5)$		8.990	3.209	0.470	8.990	3.199	0.491	8.986	3.225	0.468
			-0.010	0.009	-0.030	-0.010	-0.001	-0.001	-0.014	0.025	-0.032
			0.031	0.023	0.031	0.031	0.029	0.029	0.029	0.029	0.031
50			9.011	3.191	0.523	9.005	3.187	0.487	8.965	3.208	0.506
			0.011	-0.009	0.023	0.005	-0.013	-0.013	-0.035	0.008	0.006
			0.026	0.030	0.028	0.026	0.029	0.029	0.030	0.028	0.025
80			9.012	3.187	0.508	8.997	3.186	0.511	8.977	3.208	0.493
			0.012	-0.013	0.008	-0.003	-0.014	-0.014	-0.023	0.008	-0.007
			0.031	0.028	0.027	0.030	0.027	0.027	0.030	0.028	0.023
100			8.997	3.211	0.493	8.988	3.189	0.507	8.972	3.192	0.512
			-0.003	0.011	-0.007	-0.012	-0.011	-0.011	-0.028	-0.008	0.012
			0.029	0.029	0.026	0.027	0.027	0.027	0.030	0.028	0.022
30	$(4, 5, 0.5)$		4.091	4.939	0.484	4.008	5.001	0.477	3.956	4.985	0.486
			0.091	-0.061	-0.016	0.008	0.001	0.001	-0.044	-0.015	-0.014
			0.036	0.041	0.038	0.027	0.030	0.030	0.029	0.032	0.017
50			4.092	4.947	0.495	4.029	4.998	0.493	3.954	4.992	0.505
			0.092	-0.053	-0.005	0.029	-0.002	-0.002	-0.046	-0.008	0.005
			0.036	0.027	0.030	0.025	0.030	0.030	0.025	0.028	0.019
80			4.065	4.956	0.455	4.003	4.986	0.508	3.939	4.984	0.504
			0.065	-0.044	-0.045	0.003	-0.014	-0.014	-0.061	-0.016	0.004
			0.029	0.035	0.027	0.019	0.027	0.027	0.025	0.031	0.018
100			4.085	4.971	0.490	3.991	4.986	0.503	3.942	4.990	0.499
			0.085	-0.029	-0.010	-0.009	-0.014	-0.014	-0.058	-0.010	-0.001
			0.035	0.026	0.028	0.019	0.030	0.030	0.022	0.032	0.020

Table 5.1: Parameter estimates for TIW distribution in some scenarios with 10.000 replicates

Monte Carlo simulations allow us to compare statistical methods. Then, we are interested in verifying the performance of the estimators. For this case, we calculate in Table 5.1 the mean, bias, and mean squared error (MSE) estimates between the theoretical and empirical distribution function as the performance assessment criteria of each parameter in Moments, Maximum likelihood, and Log-Cumulants methods. 10.000 samples from the TIW distribution were generated for different parameter values $(30, 50, 80, 100)$ sample sizes. The samples were obtained by using (5.2).

The relative bias of an estimator is defined as the mean of the estimates minus the true value of the parameter, this measurement being divided by the true value of the parameter, expected that this measurement is close to zero. Therefore, we can see that the λ parameter, the LC showed better results in terms of bias and MSE compared to the others.

The mean square error, another measure used in this work, is the mean of the difference between the estimator value and the squared parameter. The ideal would be to have a MSE equal to zero, however, in practice, this is unfeasible. Overall, we look for estimators that have a relative bias and a MSE close to zero.

Table 5.1 displays the estimatives for MM, ML and LC methods in different scenarios. In the first scenario for $\hat{\beta}$ and $\hat{\lambda}$ the MSE in MM decreasing but not in all sample sizes. However, in the last escenario, $\hat{\beta}$, $\hat{\eta}$ and $\hat{\lambda}$ for LC we observe that MSE decreases as the sample size increases. We can notice, in general, that all estimators achieved good results in terms of bias and MSE, close to the true parameters.

For the first scenario, $(3, 2, -0.7)$. Specifically, for β, the MM estimator together with the ML estimator obtained better results in terms of bias in absolute value. For the λ parameter, in the first scenario, the LC estimator presented better results, obtaining a lower MSE in all scenarios compared to MM and ML as the sample size increases. For the η parameter, the ML estimator excelled in terms of bias in all sample sizes.

The absolute bias decreases with increasing n for $\hat{\beta}$ and $\hat{\lambda}$ in MM and LC, respectively for the last scenario. But in the majority cases the bias decreases. For the three scenarios we can observe in the general form that the absolute bias and MSE decrease when the sample size is increasing, showing good performance in all methods.

Complementary to numerical results are shown in Figures 5.2 and 5.3, where the graphics corresponds to sd and MSE parameter values fixing λ in each scenario. The MSE for LC Figure 5.3(b) and (c) in second and third scenarios decreases when the sample size increases, however, in the last scenario in a general form the MSE decreases but not as we wish. It can be seen that we recommend using the LC, as it presented better results than the MM and ML in most scenarios. For example, in Figure 5.2 (a) the LC presented a sd smaller in relation to the MM and ML, which means that the estimated values are closer to the mean than the other estimators.

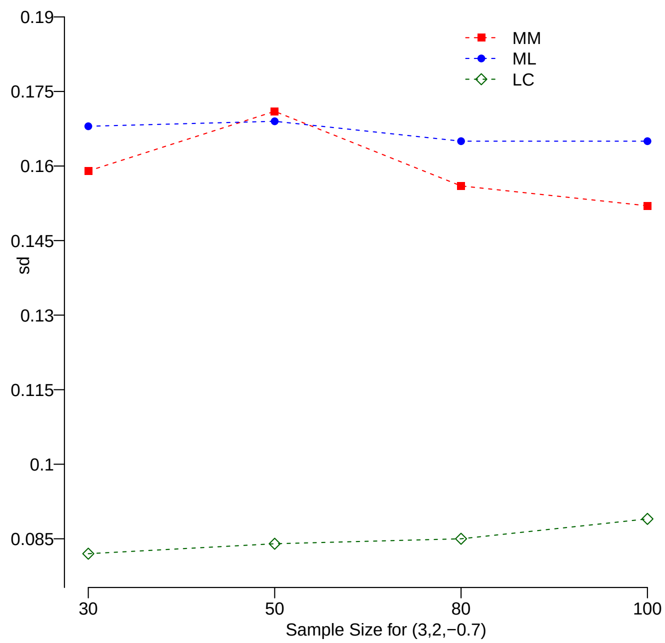
In Figure 5.3 (a), we can see that the MM estimator is the one with the lowest MSE as the sample size increases. In Figure 5.3 (c), we can see that at the sample size ($n = 80$), the LC and ML estimator had almost the same performance (MSE). In general, we can see that the results for LC present good performance, showing our proposals have furnished good results.

- **Algorithm**

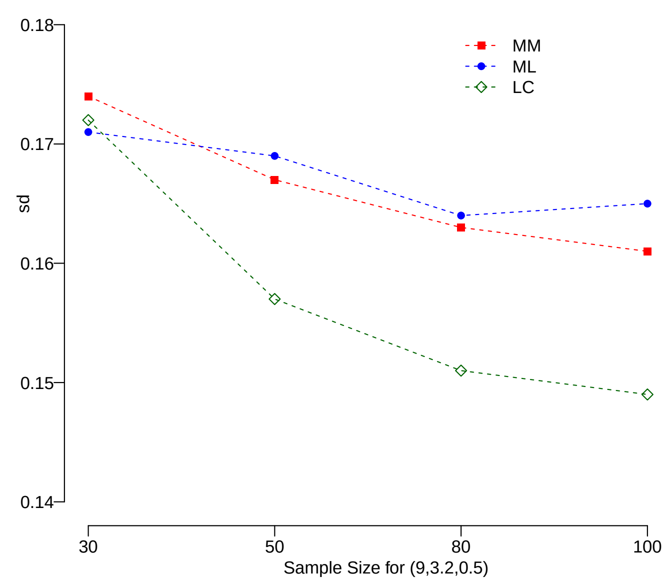
    1. Set $(\beta, \eta, \lambda) \in \{(3, 2, -0.7); (9, 3.2, 0.5); (4, 5, 0.5)\}$;
    2. Generate a random sample $x = (x_1, x_2, \ldots, x_n)$ from the TIW distribution in (β, η, λ) with sample sizes 30, 50 80 and 100;
    3. Estimate (β, η, λ) by MM, ML and LC;
    4. Calculate: For $\varepsilon > 0$, if $\left|\beta - \hat{\beta}\right| < \varepsilon$ and $|\eta - \hat{\eta}| < \varepsilon$ and $\left|\lambda - \hat{\lambda}\right| < \varepsilon$, take $(\hat{\beta}, \hat{\eta}, \hat{\lambda})$, otherwise do

it again;

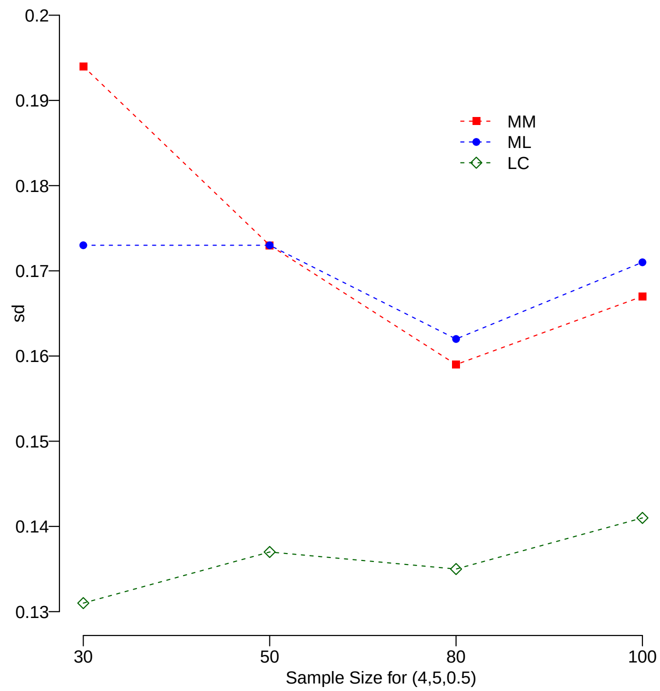
5. Generate 10,000 Monte Carlo replicates, each one with sample sizes 30, 50 80 and 100;
6. Take the Mean, Bias and MSE of those estimates.


MM
ML
LC
sd
0.19
0.175
0.16
0.145
0.13
0.115
0.1
0.085
30
50
80
100
Sample Size for (3,2,−0.7)

(a)

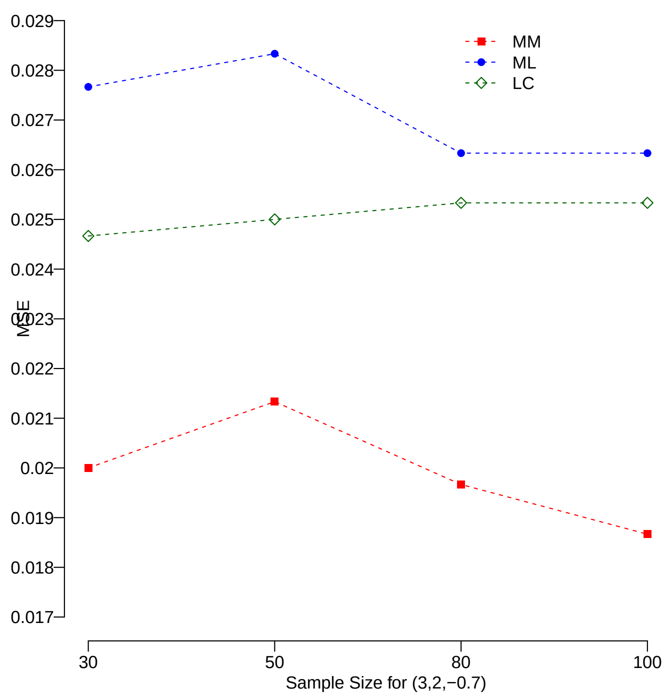

MM
ML
LC
sd
0.18
0.17
0.16
0.15
0.14
30
50
80
100
Sample Size for (9,3.2,0.5)

(b)

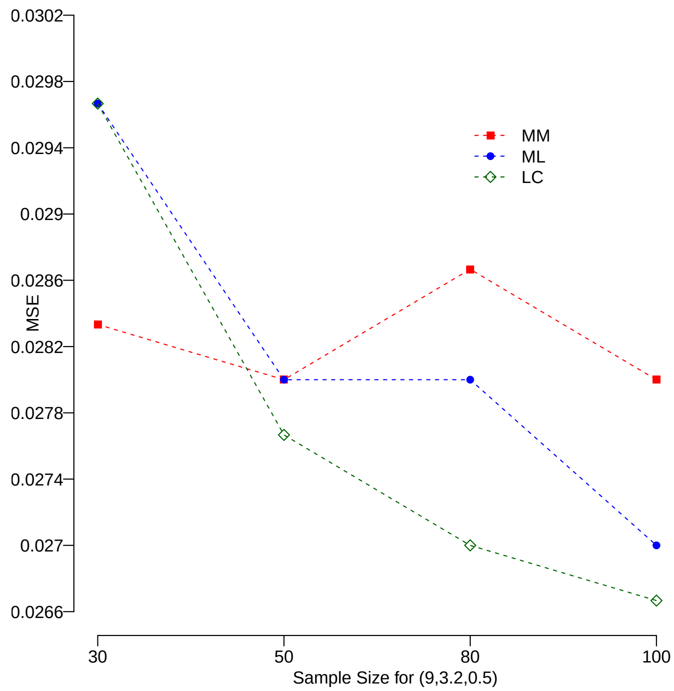


(c)

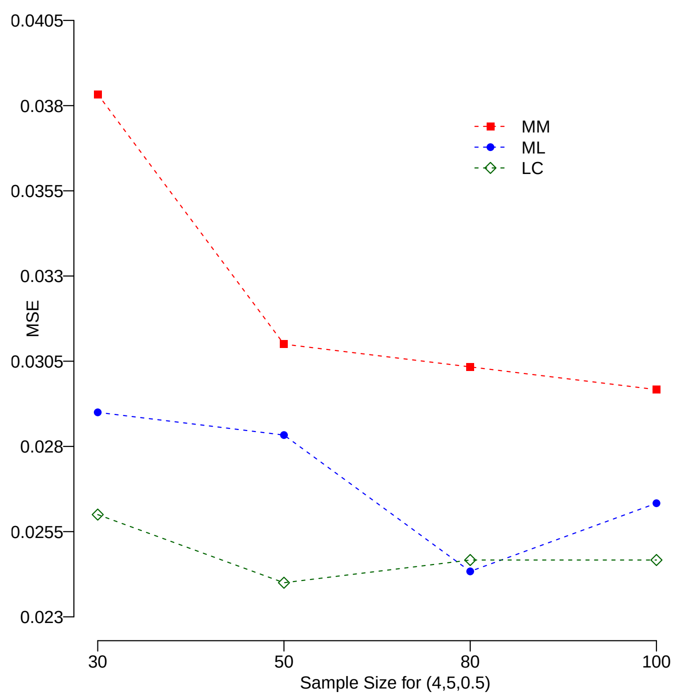
Figure 5.2: Standar deviation of the estimates of λ parameter



(a)



(b)



(c)

Figure 5.3: Mean Squared Error of the estimates of the parameters

Parameters	Sample	1%			5%			10%		
		MM	ML	LC	MM	ML	LC	MM	ML	LC
$(\beta = 3, \eta = 2, \lambda = -0.7)$	30	99.41	99.45	99.59	95.24	95.54	95.31	90.66	91.49	94.61
	50	99.53	99.61	99.76	95.32	95.50	95.59	91.14	91.06	92.78
	80	99.64	99.72	99.88	95.37	95.82	95.59	91.28	91.53	91.00
	100	99.69	99.77	99.90	95.57	95.65	93.12	91.57	90.81	93.12
$(\beta = 9, \eta = 3.2, \lambda = 0.5)$		99.84	99.81	99.78	96.35	94.20	96.02	88.40	91.40	89.04
		99.95	99.87	99.92	96.81	94.31	96.88	87.88	91.50	89.86
		99.91	99.95	99.94	96.51	94.35	96.13	87.90	91.55	90.30
		99.94	99.96	99.94	96.90	94.70	95.90	88.10	91.80	90.25
$(\beta = 4, \eta = 5, \lambda = 0.5)$		99.72	99.76	99.82	95.84	95.25	96.05	91.84	94.55	94.77
		99.84	99.87	99.82	96.01	95.28	96.67	92.33	94.56	92.34
		99.92	99.89	99.92	95.71	95.26	96.53	94.61	94.29	92.10
		99.92	99.89	99.92	95.71	95.20	96.20	94.70	93.00	90.20

Table 5.2: Estimated power for TIW distribution

For each scenario, we calculate MM, ML and LC based on the parameters, and then we calculate the count of the required p-value according to Table 5.2 to obtain a power of 90 at 0.10 level, 95 at 0.05 level and 99 at 0.01 level.

Power here is the rejection rate of H_0 at the $1\%, 5\%, 10\%$ significance level after independent simulations. The experiment was conducted with 10.000 replicates with different samples sizes in each scenario parameters. The first scenario in Table 5.2, shows that for sample size 30 in all significance levels the MM method was better than ML and LC. MM is better in this scenario for the 5% and 10% level. For the 1%, the best is the LC, as it is closer to 99%.

For 50 and 100 sample size with 1% and 5% level in the MM were close to the power of each level, and for 10% level was the ML. Already for sample size 80, LC at 10% level showed better results.

For the first scenario, it is noteworthy that at the 1% level, the LC was the one with the greatest power, as it is closer to 99% in all sample sizes. For the 5% and 10% levels, considering the sample size 30 , the MM showed greater power. As for the sample size $n = 100$, in the first scenario, considering the 10% level, the ML showed better power, as it is closer to 90%.

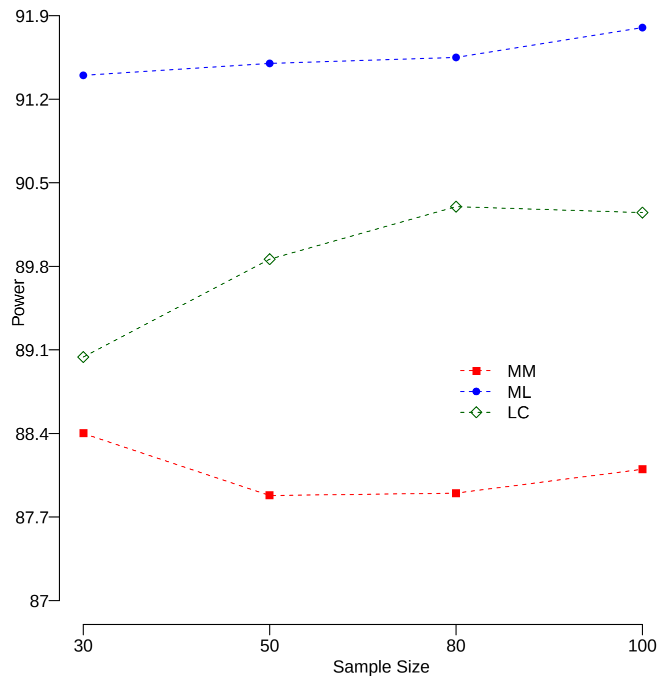
For the second scenario $(\beta = 9, \eta = 3.2, \lambda = 0.5)$, it can be seen that for the 5% and 10% levels, the LC presented better results, followed by the ML and finally the MM, as evidenced by if so as the sample size increases. Now, we can see the following in Figure 5.4, 5.5 and 5.6. For 10% level in all sample sizes LC obtained better results in the power. Also, when the sample size increases, in this case the statistical power stabilizer in a little bit greater than 90. For 5% level and sample size 100, ML and LC present closer values to the power. Finally, for 1% level, MM and LC with sample size 80 and 100 are the ones with the best results.

For the last $(\beta = 4, \eta = 5, \lambda = 0.5)$ scenario, in 10% level, for all samples sizes in LC obtain best results. And for 5% level, ML shows good results. For 1% level, with 80 and 100 sample sizes, MM and LC displays same results 99.92 for 30 and 50 sample sizes, MM and LC provide power which are close to 99.

This study was developed with the `optim` and `dfsane` functions in the `BB` package (Varadhan and Gilbert, 2010), avaliable in the Software R.

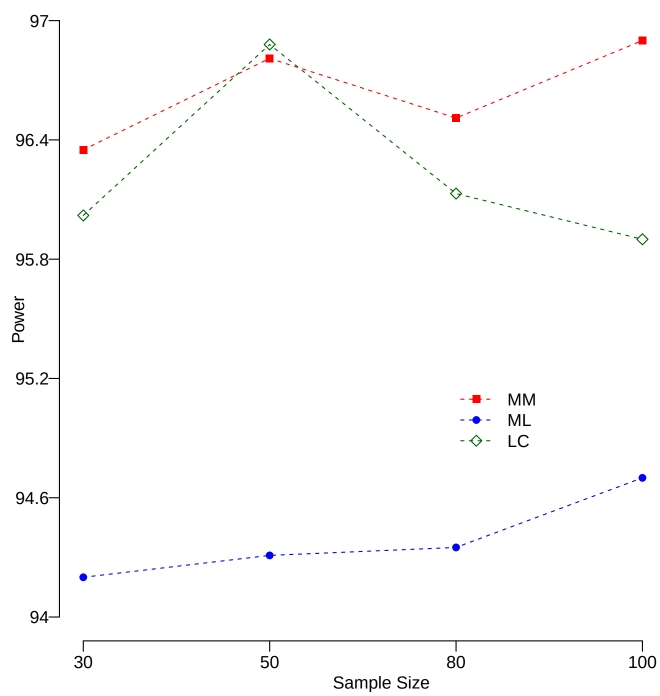
- **Algorithm Power of the test**

1. Set $(\beta, \eta, \lambda) \in \{(3, 2, -0.7); (9, 3.2, 0.5); (4, 5, 0.5)\}$;
2. Generate a random sample $x = (x_1, x_2, \ldots, x_n)$ from the TIW distribution in (β, η, λ) with sample sizes 30, 50 80 and 100;
3. Estimate the parameters with MM, ML, LC methods;
4. Calculate: For $\varepsilon > 0$, if $\left|\beta - \hat{\beta}\right| < \varepsilon$ and $|\eta - \hat{\eta}| < \varepsilon$ and $\left|\lambda - \hat{\lambda}\right| < \varepsilon$, take $(\hat{\beta}, \hat{\eta}, \hat{\lambda})$, otherwise do it again;
5. If $p-$value $> \{0.90; 0.95; 0.99\}$, $\left|\beta - \hat{\beta}\right| < \varepsilon$ and $|\eta - \hat{\eta}| < \varepsilon$ and $\left|\lambda - \hat{\lambda}\right| < \varepsilon$, take $(\hat{\beta}, \hat{\eta}, \hat{\lambda})$;
6. Take the Mean of those estimates.



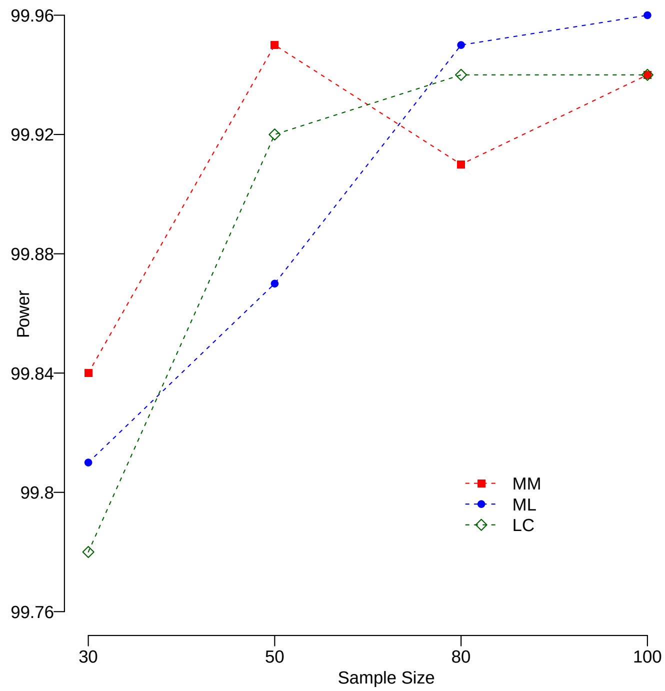
Level 10%

Figure 5.4



Level 5%

Figure 5.5



Level 1%

Figure 5.6

Parameters	Sample	1%			5%			10%		
		T^2	W	KS	T^2	W	KS	T^2	W	KS
$(\beta = 3, \eta = 2, \lambda = -0.7)$	30	99.62	99.02	98.60	96.05	95.57	96.51	91.03	92.35	94.89
	50	99.75	98.99	98.95	96.13	95.57	96.76	91.51	92.14	94.58
	80	99.87	98.98	99.22	95.87	95.51	96.88	91.50	92.20	94.57
	100	99.90	98.99	99.33	95.40	95.50	95.55	91.50	92.14	94.50
$(\beta = 9, \eta = 3.2, \lambda = 0.5)$		99.70	99.03	98.60	95.48	95.57	96.51	92.14	92.35	94.89
		99.71	98.99	98.95	96.10	95.57	96.76	90.83	92.14	94.57
		99.85	98.99	99.23	96.76	95.51	96.88	90.80	92.21	94.58
		99.89	99.00	99.34	95.79	95.51	96.55	90.50	92.15	94.55
$(\beta = 4, \eta = 5, \lambda = 0.5)$		99.79	99.03	98.61	95.19	95.57	96.51	93.17	92.35	94.89
		99.86	99.00	98.96	95.20	95.57	96.76	92.70	92.14	94.40
		99.93	98.99	99.23	96.88	95.51	96.88	92.30	92.21	94.50
		99.92	99.00	99.34	95.15	95.51	96.55	92.28	92.14	94.51

Table 5.3: Estimated power Goodness-of-Fit measures for TIW distribution

In this study, we would hope that the simulated power for each of these three scenarios would be close to 90, 95 and 99. Table 5.3 displays the estimated power for GoF measures of the T^2, W, KS statistics. Here, we generate samples from TIW and we do a Monte Carlo simulation with ten thousand replicas each one. With $\alpha = 1\%, 5\%, 10\%$ levels and sample sizes $n = 30, 50, 80, 100$, we will analyse the results.

We use the Kolmogorov-Smirnov and Cramér-von Mises statistics criteria to assess the appropriateness of the model. The least value of these measures may indicate better fit. The performance of T^2 measure, is compared with the W and KS GoF measures.

From Table 5.3, the first scenario, with 10% level and all samples sizes, the T^2 shows best perform than others. For the 1% and 5% levels, W test presents values closer to 99 and 95, respectively. It was more powerful in all sample sizes.

In $(\beta = 9, \eta = 3.2, \lambda = 0.5)$ scenario, with 1% and 5% levels in all samples sizes W test outperform the others. For this same scenario, the maximum value of the estimating power was verified, with $n = 100$, for the T^2 test, at the level of 10%. In general, it is noted that T^2 was more powerful at the 10% level.

In the last scenario, for the 1% and 10% levels, W test displays better results. For sample sizes 50 and 100 and for 5% level, the T^2 tool, in general way shows good results. It can also be noted that, for a sample size, $n = 50$, W excelled in terms of power in the scenarios for 1% and 10%.

This study was carried out by using the `BB` package (Varadhan and Gilbert, 2010), also by using `optim` and `dfsane` functions avaliables in the software R Core Team.

- **Algorithm**

  1. Let $x = (x_1, x_2, \ldots, x_n)$ a random sample following the TIW distribution;
  2. Write the CDF of TIW to use the statistics of item 4. below;
  3. Estimate the parameters with LC method to calculate T^2 statistic;
  4. Use the Kolmogorov Smirnov and Cramér-Vom Mises criterion ;
  5. Calculate: For $\varepsilon > 0$, if $\left|\beta - \hat{\beta}\right| < \varepsilon$ and $|\eta - \hat{\eta}| < \varepsilon$ and $\left|\lambda - \hat{\lambda}\right| < \varepsilon$, take $(\hat{\beta}, \hat{\eta}, \hat{\lambda})$, otherwise do it again.

5.5 Analysis in real dataset

In this section we provide an analysis from real datasets in order to evaluate the performance of the estimators. In the present study we will be using the next three datasets:

- DATASET 1: The data are taken from Aarset (1987) and also reported in Khan and King (2014), refers to the failure times of fifty devices put on life test at time zero. This dataset is known to have a bathtubshaped hazard rate. The observations (in weeks) are

0.1	0.2	1.0	1.0	1.0	1.0	1.0	2.0	3.0	6.0
7.0	11.0	12.0	18.0	18.0	18.0	18.0	18.0	21.0	32.0
36.0	40.0	45.0	45.0	47.0	50.0	55.0	60.0	63.0	63.0
67.0	67.0	67.0	67.0	72.0	75.0	79.0	82.0	82.0	83.0
84.0	84.0	84.0	85.0	85.0	85.0	85.0	85.0	86.0	86.0

Table 5.4: DATASET 1. Life time of 50 devices

- DATASET 2: The data in Lee and Wang (2003), represents a set of remission times (in months) of 128 bladder cancer patients reported.

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63	0.20	2.23	3.52	4.98	6.97
9.02	13.29	0.40	2.26	3.57	5.06	7.09	9.22	13.80	25.74	0.50	2.46	3.64
5.09	7.26	9.47	14.24	25.82	0.51	2.54	3.70	5.17	7.28	9.74	14.76	26.31
0.81	2.62	3.82	5.32	7.32	10.06	14.77	32.15	2.64	3.88	5.32	7.39	10.34
14.83	34.26	0.90	2.69	4.18	5.34	7.59	10.66	15.96	36.66	1.05	2.69	4.23
5.41	7.62	10.75	16.62	43.01	1.19	2.75	4.26	5.41	7.63	17.12	46.12	1.26
2.83	4.33	5.49	7.66	11.25	17.14	79.05	1.35	2.87	5.62	7.87	11.64	17.36
1.40	3.02	4.34	5.71	7.93	11.79	18.10	1.46	4.40	5.85	8.26	11.98	19.13
1.76	3.25	4.50	6.25	8.37	12.02	2.02	13.31	4.51	6.54	8.53	12.03	20.28
2.02	3.36	6.76	12.07	21.73	2.07	3.36	6.93	8.65	12.63	22.69		

Table 5.5: DATASET 2. Remission times of 128 bladder cancer patients

- DATASET 3: The data are the time between failures (thousands of hours) of 23 secondary reactor pumps installed in RSG-GAS reactor in Suprawhardana and Prayoto (1999) and also Bebbington et al. (2007).

2.160	0.746	0.402	0.954	0.491	6.560	4.992	0.347	0.150
0.358	0.101	1.359	3.465	1.060	0.614	1.921	4.082	0.199
0.605	0.273	0.070	0.062	5.320				

Table 5.6: DATASET 3. Time between failures of secondary reactor pumps

For the above datasets, in Table 5.7 we show some descriptive statistics for each one. Coefficient of Variation (CV) values indicate greater levels of dispersion in data around the mean. Related to asymmetry and kurtosis in dataset 1, they indicate left tail and smooth distribution. In dataset 2, for the mean, median and moda values, we have skewed left distribution. Standard Deviation (SD) in dataset 3, observations indicate low dispersed relation to the mean.

Dataset	**Min**	**Max**	**Mean**	**Median**	**Moda**	**SD**	**Asymmetry**	**Kurtosis**	**CV**
1	0.10	86.00	45.69	48.50	1.00	32.84	-0.13	-1.64	71.87
2	0.08	79.05	9.37	6.39	5.32	10.51	3.29	18.48	112.20
3	0.06	6.56	1.55	0.61	0.75	1.97	1.29	0.18	127.10

Table 5.7: Descriptive statistics for datasets

	n	Parameters	Estimatives	Mean	Bias	Var	Sd
Bootstrap	30	β	2.4686	2.566	0.830	0.378	0.615
		η	0.0015	0.013	−0.009	0.005	0.068
		λ	−0.7964	−1.021	−0.421	0.127	0.356
	50		2.4686	2.776	-0.280	0.090	0.300
			0.0015	0.001	0.00	0.000	0.001
			−0.7964	-0.963	0.297	0.039	0.198
	80		2.4686	2.789	-0.139	0.068	0.261
			0.0015	0.001	0.000	0.000	0.001
			−0.7964	-0.948	-0.021	0.024	0.154
	100		2.4686	2.774	-0.460	0.050	0.223
			0.0015	0.001	0.001	0.000	0.001
			−0.7964	-0.935	0.343	0.016	0.127
Jackknife	30	β	2.4686	2.573	0.104	0.392	0.6258
		η	0.0015	0.0147	0.0132	0.0052	0.0726
		λ	−0.7964	−1.014	−0.218	0.1232	0.3511
	50		2.4686	2.777	0.309	0.089	0.299
			0.0015	0.0011	-0.0003	0.000	0.0013
			−0.7964	-0.9602	-0.163	0.0392	0.198
	80		2.4686	2.788	-0.405	0.067	0.259
			0.0015	0.0009	0.0007	0.000	0.0008
			−0.7964	-0.951	0.011	0.0236	0.1538
	100		2.4686	2.772	0.258	0.0496	0.2228
			0.0015	0.0009	-0.0005	0.0000	0.00060
			−0.7964	-0.932	-0.135	0.0161	0.127

Table 5.8: Dataset 1: Bootstrap and Jackknife

	n	Parameters	Estimatives	Mean	Bias	Var	Sd
Bootstrap	30	β	2.44785119	2.403	-0.211	0.581	0.763
		η	0.03419	0.089	0.068	0.008	0.088
		λ	-0.53342	-0.899	0.025	0.165	0.406
	50		2.44785119	2.474	-0.245	0.333	0.577
			0.03419	0.058	0.026	0.002	0.045
			-0.53342	-0.817	-0.265	0.110	0.332
	80		2.44785119	2.495	0.468	0.207	0.455
			0.03419	0.050	-0.043	0.001	0.032
			-0.53342	-0.796	-0.211	0.088	0.297
	100		2.44785119	2.527	-0.104	0.114	0.338
			0.03419	0.043	0.013	0.001	0.023
			-0.53342	-0.789	-0.036	0.071	0.267
Jackknife	30	β	2.44785119	2.3962	-0.051	0.5364	0.7324
		η	0.03419	0.0951	0.0610	0.0143	0.1197
		λ	-0.53342	-0.878	-0.344	0.1228	0.3504
	50		2.44785119	2.5056	-0.0578	0.3608	0.600
			0.03419	0.056	0.0224	0.0022	0.0469
			-0.53342	-0.7912	-0.2578	0.0578	0.2405
	80		2.44785119	2.512559	-0.0647	0.2514	0.5014
			0.03419	0.0494	0.0152	0.0010	0.0326
			-0.53342	-0.7996	-0.2662	0.0622	0.2494
	100		2.44785119	2.59383	0.145	0.1902	0.4362
			0.03419	0.0422	0.0080	0.00056	0.02373
			-0.53342	-0.8795	-0.3461	0.0612	0.2601

Table 5.9: Dataset 2: Bootstrap Jackknife

	n	Parameters	Estimatives	Mean	Bias	Var	Sd
	30	β	1.4206	1.376	-0.483	0.115	0.339
		η	4.595	4.040	1.591	2.259	1.503
		λ	-0.580	-0.339	0.249	0.324	0.569
	50		1.4206	1.381	0.133	0.085	0.291
			4.595	4.203	-1.862	1.792	1.339
			-0.580	-0.392	0.197	0.262	0.512
Bootstrap							
	80		1.4206	1.392	-0.264	0.051	0.225
			4.595	4.397	0.619	1.170	1.081
			-0.580	-0.453	0.127	0.182	0.427
	100		1.4206	1.408	-0.169	0.022	0.150
			4.595	4.539	0.050	0.536	0.732
			-0.580	-0.514	0.057	0.097	0.311
	30	β	1.4206	1.381	-0.039	0.1179	0.3434
		η	4.595	4.0078	-0.5877	2.2240	1.491
		λ	-0.580	-0.336	0.2441	0.3255	0.5705
	50		1.4206	1.3810	-0.0396	0.0822	0.2867
			4.595	4.2336	-0.361	1.756	1.325
			-0.580	-0.3956	0.1846	0.2578	0.5077
Jackknife							
	80		1.4206	1.3909	-0.0297	0.0532	0.2307
			4.595	4.385	-0.2096	1.2030	1.0968
			-0.580	-0.4477	0.1325	0.1896	0.4354
	100		1.4206	1.4119	-0.008	0.01995	0.1412
			4.595	4.5546	-0.040	0.4997	0.7069
			-0.580	-0.5194	0.0608	0.08827	0.2971

Table 5.10: Dataset 3: Bootstrap Jackknife

Tables 5.8, 5.9 and 5.10 show the results of Jackknife and bootstrap estimates. We can see that estimates of both methods are similar to each other and the original population whose objective is to estimate the variability of an estimator resampled from the observed sample itself. The corresponding estimatives for β, η, λ are given. The mean, bias, variance and standar deviation are presented for each dataset.

In dataset 2, the jackknife estimates for β in sample zise 100 the corresponding estimated standard deviation is the smaller value 0.02373.

We have the smallest values for the third dataset bias and standar deviation in Bootstrap for β and λ.

Given the results presented in table 5.10 the Jackknife method proved to be more efficient than the bootstrap method because it showed the smallest bias for the β, η and λ parameters for sample sizes $n = 30, 50, 80$ For $n = 100$ the Bootstrap method for the λ parameter resulted in a smaller bias of 0.057 versus 0.0608 for the Jackknife. Table 5.10 shows good results that are closer in the two methods. For both methods, the bias in the shape parameter η decreases as the sample size increases.

The bias for λ transmuting parameter in all datasets, decreases as the sample size increases in almost all the cases The TIW distribution approximately provides an adequate fit for the data.

5.6 Concluding

In this chapter, a simulation study was done to verify the performance of the parameters. With different scenarios, from Monte Carlo simulations, the behavior of the estimates from TIW distribution were evaluated by Moments, Maximum Likelihood and Log-Cumulants methods. The results obtained showed that the parameter estimates present good results in terms of bias and MSE. We use sd and MSE to illustrate the performance of the parameters. Also, Goodness of Fit measures are provided for TIW distribution. Cramér-von Mises and Kolmogorov Smirnov are compare with T^2 statistic.

Chapter 6

Concluding and Future Works

As a mentioned, the first article (Chapter 3) is independent of the second article (Chapter 4) and third (Chapter 5), but we trying to give new results in which one. The contributions are linked to the objectives, therefore we mention as follow:

- To give new results in stochastic comparisons.
- To use the majorization theory and order statistics to do new theorems and lemmas.
- To use the Exponentiated Generalized class to compare parallel systems by the largest statistics.
- To propose a new measures of Goodness-of-Fit for the Transmuted Inverse Weibull distribution.
- To do a computational study with the Transmuted Inverse Weibull distribution.
- To compare the Moments, Maximum Likelihood and Log-Cumulantes methods.
- To applied datasets in the survival analysis context.

Recapitulating, some different new results are given to accomplish the development of the thesis:

In Chapter 3 we have comparisons stochastics by using the Exponentiated Generalized class with so many examples and counterexamples. We use the majorization theory and stochastic order to do new theorems and lemmas.

In Chapter 4, the focus is on the Mellin Transform and goodness of fit with the Tansmuted Inverse Weibull. The T^2 statistic is used to choose the best fit to the data.

The next Chapter is following the same line of the second article (Chapter 4), we are doing an excellent and complete applied computational study of simulation by using Monte Carlo and bootstrap methods, we use the

Maximun Likelihood, Moments and Log-Cumulants methods to verify the best performance. Also measure the power of the test and to choose the best method of estimation in survival analysis field.

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